



Revisiting independence and stochastic dominance for compound lotteries

Alexander Zimmer¹

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¹ Department of Economics and Econometrics, University of Johannesburg

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Alexander Zimmer[†]

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Abstract

We establish mathematical equivalence between independence of irrelevant alternatives and monotonicity with respect to first order stochastic dominance. This formal equivalence result between the two principles is obtained under two key conditions. Firstly, for all $m \in \mathbb{N}$, each principle is defined on the domain of compound lotteries with compoundness level m . Secondly, the standard concept of reduction of compound lotteries applies.

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[†]Department of Economics and Econometrics, University of Johannesburg, PO Box 524, Auckland Park, 2006, South Africa. E-mail: zimmer@bigfoot.com

1 Introduction

In the theory of decision making under risk *independence of irrelevant alternatives* serves as key assumption for deriving von Neumann and Morgenstern's (1947) representation of preferences over lotteries by an expected utility functional (cf. Fishburn 1988). However, since many real-life decision makers persistently violate independence of irrelevant alternatives (e.g., Allais 1979), alternative models of decision making under risk (for surveys on this vast literature see, e.g., Schmidt 2004; Starmer 2000; Sugden 2004) typically weaken the independence assumption while they additionally impose *monotonicity with respect to first order stochastic dominance*. Whereas the normative appeal of independence of irrelevant alternatives is controversially discussed in the literature, researchers commonly agree upon that monotonicity with respect to first order stochastic dominance must be obeyed by rational decision makers. As our main finding we establish mathematical equivalence of these two - seemingly different - principles under two conditions. As a first condition we assume that each principle is defined, for all $m \in \mathbb{N}$, on the domain of compound lotteries with compoundness level m . As a second condition we assume that the standard principle of reduction of compound lotteries applies without restriction. Referring to the development of descriptive decision theories as a reaction to the Allais paradoxes, Duncan Luce (2000) writes:

“Some theorists [...] have to a degree abandoned independence, but have continued to devise theories on the assumption that the lotteries are well modeled as random variables, and so they accept the (automatic) reduction of compound gambles. Others of us have interpreted the body of evidence as favoring consequence monotonicity and as making dubious at a descriptive level the probability reduction principle built into the random variable notation [...]” (p. 47)

This note's equivalence result once more emphasizes that the reduction principle for compound lotteries is not merely an innocuous assumption but rather plays an important (and typically hidden) role in the normative interpretation of decision theoretic axioms.

2 Notation and preliminaries

Let L^0 denote some finite set of deterministic outcomes and iteratively define for all $m \in \mathbb{N}$:

$$L^m = \left\{ (\alpha_1, s_1; \dots; \alpha_n, s_n) \mid \alpha_1, \dots, \alpha_n \in \mathbb{R}_+, \sum_{i=1}^n \alpha_i = 1 \text{ and } \exists j \in \{1, \dots, n\} \text{ s.t. } s_j \in L^{m-1} \right\}.$$

We interpret the elements of L^m as *compound lotteries of level m* , that is, there is at least one compound lottery of level $m - 1$ in the support of a compound lottery in L^m . According to this interpretation, the set $\mathbb{L} = \bigcup_{m \in \mathbb{N}} L^m$ collects all compound lotteries without any upper bound for their compoundness level m . As standard notational conventions we introduce *order-irrelevance of entries*, i.e.,

$$(\dots; \alpha, s; \dots; \beta, t; \dots) = (\dots; \beta, t; \dots; \alpha, s; \dots)$$

and *distribution of weights for identical entries*, i.e.,

$$(\dots; \alpha, s; \dots; \beta, s; \dots) = (\dots; (\alpha + \beta), s; \dots).$$

Definition. *The preference relation \succeq on \mathbb{L} satisfies complete ordering if and only if the following three assumptions are fulfilled:*

(i) *There exists an asymmetric, non-reflexive, and transitive binary relation \succ over compound lotteries in \mathbb{L} .*

(ii) *$s \sim t$ if and only if not $s \succ t$ and not $t \succ s$.*

(iii) *$s \succeq t$ if and only if $s \succ t$ or $s \sim t$.*

Definition. *The preference relation \succeq on \mathbb{L} satisfies reduction of compound lotteries if and only if, for all $m \in \mathbb{N}$, the following two assumptions are fulfilled (where $\beta_k \alpha_k$ denotes the real number resulting from the multiplication of β_k with α_k , $k \in \{1, \dots, n\}$):*

(i) *If*

$$\begin{aligned} (\beta_1, (\alpha_1, s_1; \dots; \alpha_n, s_n); \dots) &\in L^{m+1} \\ (\beta_1 \alpha_1, s_1; \dots; \beta_1 \alpha_n, s_n; \dots) &\in L^m \end{aligned}$$

then

$$(\beta_1, (\alpha_1, s_1; \dots; \alpha_n, s_n); \dots) \sim ((\beta_1 \alpha_1, s_1; \dots; \beta_1 \alpha_n, s_n); \dots).$$

(ii) *If $(1, s) \in L^{m+1}$ and $s \in L^m$ then $(1, s) \sim s$.*

3 Result

Consider a compound lottery

$$s = (\alpha_1, s_1; \dots; \alpha_n, s_n) \in L^{m+1}$$

such that $s_{k+1} \succeq s_k$ for all $k \in \{1, \dots, n - 1\}$. The *cumulative distribution function* of s with respect to \succeq , $F[s] : L^m \rightarrow [0, 1]$, is defined as

$$\begin{aligned}
F[s](x) &= 0 \text{ for all } x \in L^m \text{ such that } s_1 \succ x, \\
F[s](x) &= \sum_{j=1}^k \alpha_j \text{ for all } x \in L^m \text{ such that } s_{k+1} \succ x \succeq s_k, \\
F[s](x) &= 1 \text{ for all } x \in L^m \text{ such that } x \succeq s_n.
\end{aligned}$$

We say s strictly dominates t with respect to first-order stochastic dominance, denoted $s \succ_F t$, if and only if $F[s](x) \leq F[t](x)$ for all $x \in L^m$, and $F[s](x) < F[t](x)$ for some $x \in L^m$. Analogously, $s \sim_F t$ if and only if $F[s](x) = F[t](x)$ for all $x \in L^m$.

Definition. *The preference relation \succeq on \mathbb{L} satisfies monotonicity with respect to first-order stochastic dominance with support on L^m if and only if, for all $s, t \in L^{m+1}$, $s \succ_F t$ implies $s \succ t$, and $s \sim_F t$ implies $s \sim t$.*

Definition. *The preference relation \succeq on \mathbb{L} satisfies independence of irrelevant alternatives with support on L^m if and only if, for all compound lotteries $s, t, r \in L^m$,*

$$\begin{aligned}
s \succ t &\Rightarrow (\alpha, s; (1 - \alpha), r) \succ (\alpha, t; (1 - \alpha), r) \\
s \sim t &\Rightarrow (\alpha, s; (1 - \alpha), r) \sim (\alpha, t; (1 - \alpha), r)
\end{aligned}$$

Proposition. *Suppose that the preference relation \succeq on \mathbb{L} satisfies complete ordering and reduction of compound lotteries. Then the following two statements are equivalent:*

- (i) *For all $m \in \mathbb{N}$, \succeq satisfies independence of irrelevant alternatives with support on L^m .*
- (ii) *For all $m \in \mathbb{N}$, \succeq satisfies monotonicity with respect to first order stochastic dominance with support on L^m .*

4 Proof

At first we introduce an alternative dominance relation, denoted \succeq_E , for compound lotteries and prove - by a lemma - equivalence between \succeq_E and \succeq_F .

Definition. For two compound lotteries $s, t \in L^{m+1}$ such that

$$\begin{aligned} s &= (\alpha_1, s_1; \alpha_2, s_2; \dots; \alpha_n, s_n) \\ t &= (\alpha_1, t_1; \alpha_2, t_2; \dots; \alpha_n, t_n) \end{aligned}$$

we write $s \succeq_E t$ if and only if $s_k \succeq t_k$ for all $k \in \{1, \dots, n\}$. Furthermore, $s \sim_E t$, if and only if $s_k \sim t_k$ for all $k \in \{1, \dots, n\}$.

Remark. Observe that the dominance relation \succeq_E is equivalent to so-called *consequence monotonicity* (Definition 2.3.1. in Luce 2000) whenever transitivity holds and the decision maker's uncertainty about the occurrence of events is resolved by an additive probability measure.

Lemma: Suppose that \succeq is a complete ordering. For any pair of compound lotteries

$$(\alpha_1, s_1; \dots; \alpha_l, s_l) \in L^{m+1} \tag{1a}$$

$$(\beta_1, t_1; \dots; \beta_q, t_q) \in L^{m+1} \tag{1b}$$

there exists some pair of compound lotteries

$$(\gamma_1, s'_1; \dots; \gamma_n, s'_n) \in L^{m+1} \tag{2a}$$

$$(\gamma_1, t'_1; \dots; \gamma_n, t'_n) \in L^{m+1} \tag{2b}$$

where $s'_{j+1} \succeq s'_j$ and $t'_{j+1} \succeq t'_j$, for $j \in \{1, \dots, n-1\}$, such that

$$(\alpha_1, s_1; \dots; \alpha_l, s_l) = (\gamma_1, s'_1; \dots; \gamma_n, s'_n) \tag{3a}$$

$$(\beta_1, t_1; \dots; \beta_q, t_q) = (\gamma_1, t'_1; \dots; \gamma_n, t'_n) \tag{3b}$$

and

$$(\alpha_1, s_1; \dots; \alpha_l, s_l) \succeq_F (\beta_1, t_1; \dots; \beta_q, t_q)$$

if and only if

$$(\gamma_1, s'_1; \dots; \gamma_n, s'_n) \succeq_E (\gamma_1, t'_1; \dots; \gamma_n, t'_n).$$

Proof of the Lemma

Step 1: For any given compound lotteries (1a) and (1b) we construct two compound lotteries (2a) and (2b) such that equations (3a) and (3b) are satisfied. Note that, by the notational convention *order-irrelevance of entries*, we can - without loss of generality - assume that $s_{j+1} \succeq s_j$ and $t_{j+1} \succeq t_j$, for $j \in \{1, \dots, n-1\}$.

1st iteration step. Without loss of generality suppose $\alpha_1 \leq \beta_1$ and observe that there exists a unique number i_1 such that

$$\sum_{k=1}^{i_1} \alpha_k \leq \beta_1 < \sum_{k=1}^{i_1+1} \alpha_k.$$

Construct now two lotteries $s^1, t^1 \in L^{m+1}$ such that

$$\begin{aligned} s^1 &= \left(\dots; \left(\beta_1 - \sum_{k=1}^{i_1} \alpha_k \right), s_{i_1+1}; \left(\alpha_{i_1+1} - \left(\beta_1 - \sum_{k=1}^{i_1} \alpha_k \right) \right), s_{i_1+1}; \dots \right) \\ t^1 &= \left(\dots; \left(\beta_1 - \sum_{k=1}^{i_1} \alpha_k \right), t_1; \beta_2, t_2; \dots \right) \end{aligned}$$

and observe that the first $i_1 + 1$ probability weights in s^1 and in t^1 are identical.

2nd iteration step. Focus now on the lotteries s^1, t^1 and start with the $(i_1 + 2)$ nd entry to determine whether

$$\left(\alpha_{i_1+1} - \left(\beta_1 - \sum_{k=1}^{i_1} \alpha_k \right) \right) \leq \beta_2$$

or

$$\left(\alpha_{i_1+1} - \left(\beta_1 - \sum_{k=1}^{i_1} \alpha_k \right) \right) > \beta_2.$$

Then proceed analogously to the 1st iteration step.

The above procedure determines at the k th iteration step a unique number i_k such that the first $i_k + 1$ probability weights for entries in s^k coincide with the first $i_k + 1$ probability weights for entries in t^k . Moreover, by the notational convention *distribution of weights for identical entries*, we have, for all $k \geq 1$,

$$\begin{aligned} s^k &= (\alpha_1, s_1; \dots; \alpha_l, s_l) \\ t^k &= (\beta_1, t_1; \dots; \beta_q, t_q). \end{aligned}$$

By the above procedure, any two compound lotteries s, t are transformed after a finite number of iteration steps, say M , into notationally equivalent compound lotteries s^M, t^M . Moreover, since each iteration step generates the same number of entries with identical probability weights for all entry pairs, s^M and t^M must share the same number of entries.

Step 2: In the light of step 1, for arbitrary

$$\begin{aligned} (\alpha_1, s_1; \dots; \alpha_l, s_l) &\in L^{m+1} \\ (\beta_1, t_1; \dots; \beta_q, t_q) &\in L^{m+1}, \end{aligned}$$

$$(\alpha_1, s_1; \dots; \alpha_m, s_m) \succeq_F (\beta_1, t_1; \dots; \beta_q, t_q)$$

if and only if

$$(\gamma_1, s'_1; \dots; \gamma_n, s'_n) \succeq_F (\gamma_1, t'_1; \dots; \gamma_n, t'_n). \quad (4)$$

Thus, for proving the lemma, it remains to be shown that (4) is satisfied if and only if

$$(\gamma_1, s'_1; \dots; \gamma_n, s'_n) \succeq_E (\gamma_1, t'_1; \dots; \gamma_n, t'_n). \quad (5)$$

Rewriting (4), gives, for all $x \in L^m$,

$$F[\gamma_1, s'_1; \dots; \gamma_n, s'_n](x) \leq F[\gamma_1, t'_1; \dots; \gamma_n, t'_n](x)$$

where

$$\begin{aligned} F[\gamma_1, s'_1; \dots; \gamma_n, s'_n](x) &= \sum_{k=1}^j \gamma_k \text{ such that } s'_{j+1} \succ x \succeq s'_j \\ F[\gamma_1, t'_1; \dots; \gamma_n, t'_n](x) &= \sum_{k=1}^j \gamma_k \text{ such that } t'_{j+1} \succ x \succeq t'_j. \end{aligned}$$

Our claim is trivially proved if, for all $j \in \{1, \dots, n\}$, $s_j \sim t_j$. Thus, assume that (5) is satisfied and denote by s'_j, t'_j , for $j \in \{1, \dots, n\}$, the first entries such that $s'_j \succ t'_j$ whereas $s'_k \sim t'_k$ for all $k < j$. Observe that, for all r with $t'_j \succeq x$,

$$\begin{aligned} F[\gamma_1, s'_1; \dots; \gamma_n, s'_n](x) &= \sum_{k=1}^{j-1} \gamma_k \\ &< \sum_{k=1}^{j-1} \gamma_k + \gamma_j \leq F[\gamma_1, t'_1; \dots; \gamma_n, t'_n](x) \end{aligned}$$

Repeating this argument for the remaining $r \in L^m$ shows that (4) is satisfied as well.

Assume now that (4) is satisfied and denote by $r^* \in L^m$ the compound lottery such that for all x with $r^* \succ x$,

$$F[\gamma_1, s'_1; \dots; \gamma_n, s'_n](x) = F[\gamma_1, t'_1; \dots; \gamma_n, t'_n](x)$$

whereas

$$F[\gamma_1, s'_1; \dots; \gamma_n, s'_n](r^*) < F[\gamma_1, t'_1; \dots; \gamma_n, t'_n](r^*)$$

Obviously, $s'_j \sim t'_j$ for all $j \in \{1, \dots, n\}$ such that $r^* \succ s'_j$ and $r^* \succ t'_j$. Contrary to (5), presume that $t'_{j+1} \succ s'_{j+1}$. But then

$$\begin{aligned} F[\gamma_1, s'_1; \dots; \gamma_n, s'_n](s'_{j+1}) &= \sum_{k=1}^j \gamma_k + \gamma_{j+1} \\ &> \sum_{k=1}^j \gamma_k = F[\gamma_1, t'_1; \dots; \gamma_n, t'_n](s'_{j+1}) \end{aligned}$$

A contradiction to (4). Repeating this argument finally proves our claim. \square

Proof of the Proposition. Restricted to compound lotteries $s, t \in L^{m+1}$ with two entries only, ME (=monotonicity with respect to \succeq_E) with support on L^m obviously coincides with the definition of independence with support on L^m . As a consequence, we obtain that, for arbitrary $m \in \mathbb{N}$, ME with support on L^m implies independence with support on L^m . In view of the lemma we can therefore prove the proposition by showing that, for arbitrary $m \in \mathbb{N}$, independence with support on L^{m+1} implies ME with support on L^m .

We prove this claim by induction over the number of entries in the compound lotteries in L^m . At first we show that ME with support on L^m is satisfied for all compound lotteries $s, t \in L^{m+1}$ having exactly one entry in their support.

Suppose this claim is false. Then there are compound lotteries

$$\begin{aligned} s &= (1, s') \in L^{m+1} \\ t &= (1, t') \in L^{m+1} \end{aligned}$$

such that $s' \succeq t'$ whereas $t \succ s$. By reduction of compound lotteries, $s \sim s'$ and $t \sim t'$, implying, by transitivity, $s \succeq t \succ s$. A contradiction to the ordering assumption.

Assume now that ME with support on L^m is satisfied for all compound lotteries $s, t \in L^{m+1}$ with k entries. We prove: if independence with support on L^{m+1} is satisfied, then ME with support on L^m must be satisfied for all compound lotteries $s, t \in L^{m+1}$ with $k + 1$ entries.

Suppose on the contrary that ME with support on L^m is violated for some compound lotteries $s, t \in L^{m+1}$ with $k + 1$ entries, i.e., there are compound lotteries

$$\begin{aligned} s &= (\gamma_1, s'_1; \dots; \gamma_{k+1}, s'_{k+1}) \in L^{m+1} \\ t &= (\gamma_1, t'_1; \dots; \gamma_{k+1}, t'_{k+1}) \in L^{m+1} \end{aligned}$$

such that $s_j \succeq t_j$ for all $j \in \{1, \dots, k + 1\}$ whereas $t \succ s$. By reduction of compound lotteries, $s \sim s'$ and $t \sim t'$ whereby

$$\begin{aligned} s' &= (1 - \gamma_{k+1}), \left(\frac{\gamma_1}{(1 - \gamma_{k+1})}, s'_1; \dots; \frac{\gamma_k}{(1 - \gamma_k)}, s'_k \right); \gamma_{k+1}, s'_{k+1} \in L^{m+2} \\ t' &= (1 - \gamma_{k+1}), \left(\frac{\gamma_1}{(1 - \gamma_{k+1})}, t'_1; \dots; \frac{\gamma_k}{(1 - \gamma_k)}, t'_k \right); \gamma_{k+1}, t'_{k+1} \in L^{m+2}. \end{aligned}$$

Moreover, by the induction assumption

$$\left(\frac{\gamma_1}{(1 - \gamma_{k+1})}, s'_1; \dots; \frac{\gamma_k}{(1 - \gamma_k)}, s'_k \right) \succeq \left(\frac{\gamma_1}{(1 - \gamma_{k+1})}, t'_1; \dots; \frac{\gamma_k}{(1 - \gamma_k)}, t'_k \right)$$

since $s_j \succ t_j$ for all $j \in \{1, \dots, k\}$. Because of $s_{k+1} \succ t_{k+1}$, independence with support on L^{m+1} implies $s' \succ t'$. By transitivity, $s \succ t \succ s$, contradicting the ordering assumption. $\square\square$

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