Justifiable Preferences for Freedom of Choice

Somdeb Lahiri

ABSTRACT: In this paper we say that a preference for freedom of choice is justifiable if there exists a reflexive and complete binary relation on the set of alternatives, such that one opportunity set is at least as good as a second, if and only if the there is at least one alternative from the first set which is no worse than any alternative of the two sets combined together, with respect to the binary relation on the alternatives. In keeping with the revered tradition set by von Neumann and Morgenstern we call a reflexive and complete binary relation, an abstract game (note : strictly speaking von Neumann and Morgenstern refer to the asymmetric part of a reflexive and complete binary relation as an abstract game; hence our terminology though analytically equivalent, leads to a harmless corruption of the original meaning).In this paper we obtain necessary and sufficient conditions for the justifiability of transitive and quasi transitive preferences for freedom of choice.

Indian Institute of Management,
Ahmedabad-380 015,
India,
and
ERSA
University of the Witwatersrand

1 Introduction

The dominant paradigm of welfare economics is one of choice, where an individual chooses one or more alternatives from a non empty given set of alternatives. There may or may not be an underlying preference structure with respect to which the decision is arrived at. However, welfare economics exhibits a preferential bias towards choice situations where decisions are determined by some underlying preference structure. This is what is commonly known as rational choice theory.

Some problems of decision making are naturally two stage procedures, as for instance when in a first round of interview we short list a set of candidates for a second round of interview. One way in which such procedures can be viewed is the final choices are determined by the lexicographic composition of a first round ranking followed by a second round ranking. The composition defines a binary relation on the set of alternatives.Study of such objects are available in Aizerman and Malishevsky (1986),Aizerman and Aleskerov (1995),Aleskerov (1999), Lahiri (2000) and more recently Lahiri (2001). An alternative way of viewing such two stage decision problems is to conceive the first stage as the selection of a feasible set, to which the second step choices are reflected. The first stage decision is often decided by what is known as preference for flexibility or preference for freedom of choice. Some of the very early discussions of this concept can be found in Koopmans (1964) , Kreps (1979) and Sen (1988). The problem received fresh impetus with subsequent contributions by Pattanaik and Xu (1990,1997,1998)), Klemisch – Ahlert (1993), Arrow (1994), Puppe (1995,1996), Sen (1990, 1991), to mention a few.

The most emphatic statement of the problem discussed in this paper can be traced to the paper by Arrow (1994). The standpoint that Arrow adopted was the following: There are many important situations where it makes sense to say that one opportunity set is at least as good as another if and only if there exists a reflexive ,complete and transitive binary relation, with respect to which the best element of the first set is no worse than the best element of the second. Such preferences for freedom of choice are similar to the concept of indirect utility functions of rational economic choice theory. Malishevsky (1997) provides axiomatic characterizations of such preferences for freedom of choice. It turns out that the axiomatic characterizations make significant use of a property called Monotonicity, which says that if one subset contains another subset then the bigger subset is no worse than the second. While Monotonicity is a doubtful assumption to make for a choice between being served a cup of tea in the morning and the opportunity set which includes the possibility of being “beheaded at dawn” as well, such is not the case if the possibility of being beheaded at dawn is replaced by being served coffee in the morning. The context determines the validity of an axiom and in this sense Monotonicity is no different.

In this paper we say that a preference for freedom of choice is justifiable if there exists a reflexive and complete binary relation on the set of alternatives, such that one oppurtunity set is at least as good as a second, if and only if the there is at least one alternative from the first set which is no worse than any alternative of the two sets combined together, with respect to the binary relation on the alternatives. In keeping with the revered tradition set by von Neumann and Morgenstern we call a reflexive and complete binary relation, an abstract game (note : strictly speaking von Neumann and Morgenstern refer to the asymmetric part of a reflexive and complete binary relation as an abstract game;

hence our terminology though analytically equivalent, leads to a harmless corruption of the original meaning).It turns out that if a preference for freedom of choice is justifiable, then the base relation with respect to which it is justifiable, is simply the restriction of the preference for freedom of choice, to the set of all singletons.

Our first major result is about the justifiability of transitive preferences for freedom of choice. It says that such preferences are justifiable if and only if they satisfy Monotonicity and Concordance.Concordance says that if one opportunity set is at least as desirable as a second then it should also be the case that the first opportunity set is at least as desirable as the union of the two. Since, for the case of transitive preferences for freedom of choice, our notion of justifiability coincides with that of Arrow and Malishevsky, our axiomatic characterization can throw some light on properties of indirect utility functions.

In a final section of this paper we extend the problem that was originally posed by Arrow for transitive preferences for freedom of choice to the class of all quasi transitive preferences for freedom of choice.A binary relation is quasi transitive if its asymmetric part is transitive; its symmetric part need not be transitive. It turns out that three properties, namely Strict Concatenation, Strict Monotonicity and Weak Expansion are necessary for a quasi transitive preference for freedom of choice to be justifiable. Strict Concatenation says that if there are four opportunity sets of which the first is preferred to the second and the third is preferred to the fourth, then the union of the first and third is preferred to the union of the second and fourth. Strict Monotonicity says that if one opportunity set is preferred to a second then if (a) a third set includes the first the third is preferred to the second; (b) if the third set is contained in the second then the first set is preferred to the third. Weak Expansion says that if an alternative (i.e. a singleton) is at least as desirable as two opportunity sets considered separately, then it should be at least as desirable as the opportunity set formed by the union of the two. Weak Expansion appears to be a reasonable hypothesis. It implies a property called Weak Condorcet which says that in the event that a singleton subset of a set is at least as desirable as any other singleton subset of the given set, then this singleton subset is at least as desirable as the given set. Amongst the three axioms namely Strict Concatenation , Strict Monotonicity and Weak Expansion except for the last one, the other two are extremely mild. Weak Expansion appears to be a comparatively strong assumption and it is worth investigating in future research, whether it is possible to replace it by a weaker assumption.

2 The Model

Let X be a non-empty finite set of alternatives containing at least two elements. Let [X] be the set of all non empty subsets of X. Let $\Delta(X) = \{(x,x)/x\in X\}$ and $\Delta([X]) = \{(A,A)/A\in [X]\}$. $\Delta(X)$ is called the diagonal of X and $\Delta([X])$ is called the diagonal of [X].

A binary relation R on X is said to be :

(a) reflexive, if $\Delta(X) \subset R$;
(b) complete, if given $x,y \in X$, with $x \neq y$, either $(x,y) \in R$ or $(y,x) \in R$;
(c) an abstract game if it is both reflexive and complete;

Given a binary relation R on X, let $P(R) = \{(x,y) \in R / (y,x) \notin R\}$ denote the asymmetric part of R and let $I(R) = \{(x,y) \in R / (y,x) \in R\}$ denote the symmetric part of R.

A binary relation R on X is said to be acyclic if given any positive integer n and elements x(1),…,x(n) in X : [(x(i),x(i+1)) $\in$ P(R) $\forall$ i $\in$ {1,…,n-1}] implies [(x(n),x(1)) $\notin$ P(R)].
A binary relation R on X is said to be transitive if $\forall$x,y,z $\in$ X : [(x,y),(y,z)$\in$ R] implies [(x,z)$\in$ R].

A binary relation $\Re$ on [X] is said to be :

(d) reflexive, if Δ([X]) $\subset$ $\Re$;

(e) complete, if given A,B $\in$ [X], with A $\neq$ B, either (A,B) $\in$ $\Re$ or (B,A) $\in$ $\Re$;

(f) a preference for freedom of choice (PFC) if it is both reflexive and complete;

Given a binary relation $\Re$ on [X], let P($\Re$) = {(A,B) $\in$ $\Re$ / (B,A) $\notin$ $\Re$} denote the asymmetric part of $\Re$ and let I($\Re$) = {(A,B)$\in$ $\Re$ / (B,A) $\in$ $\Re$} denote the symmetric part of $\Re$.
A binary relation $\Re$ on [X] is said to be acyclic if given any positive integer n and elements A(1),…,A(n) in [X] : [(A(i),A(i+1)) $\in$ P($\Re$) $\forall$ i $\in$ {1,…,n-1}] implies [(A(n),A(1)) $\notin$ P($\Re$)].
A binary relation $\Re$ on [X] is said to be transitive if $\forall$A,B,C $\in$ [X] : [(A,B),(B,C)$\in$ $\Re$] implies [(A,C)$\in$ $\Re$].
Given a PFC $\Re$, we define the abstract game revealed by $\Re$, to be the binary relation R($\Re$) = {(x,y) / ({x},{y}) $\in$ $\Re$}.
Given an abstract game R and A $\in$ [X], let G(A,R) = {x$\in$ A / (x,y) $\in$ R, whenever y $\in$ A}.
The following well known result can be found in Kreps [1988]:

Proposition 1: Let R be an abstract game. Then, G(A,R) $\neq\phi$ whenever A $\in$ [X] if and only if R is acyclic.
We say that a PFC $\Re$ is justifiable if there exists an abstract game R such that $\forall$A,B $\in$ [X]: [(A,B) $\in$ $\Re$] if and only if [G(A$\cup$B,R)$\cap$A $\neq\phi$].
It is easy to see that if a PFC is justifiable by R, then R = R($\Re$). Hence we have the following proposition :

Proposition 2 : If a PFC $\Re$ is justifiable by an abstract game R, then R = R($\Re$). Further, in such a situation R($\Re$) is acyclic.

Proof : The acyclicity of R($\Re$) is required by Proposition 1 and the definition of justifiability.
Q.E.D.

However, there are extremely well behaved PFC's which are not justifiable.

A PFC $\Re$ is said to be monotonic if $\forall$A,B$\in$ [X]: [B $\subset$ A] implies [(A,B) $\in$ $\Re$].
Monotonicity is a very reasonable assumption to make on a PFC. It simply says that a set which contains another set should be at least as desirable as the set it contains.

Proposition 3: Let X = {x,y,z} with x $\neq$ y $\neq$ z $\neq$ x. Then there exists a PFC $\Re$, which is transitive and monotonic, but not justifiable.

Proof : Let $\Re = \Delta([X]) \cup \{(\{x\},\{y\}),(\{y\},\{z\}),(\{x\},\{z\})\} \cup \{(A,B) \in [X]x[X] / \#(A) > \#(B)\} \cup \{(A,B) \in [X]x[X] / \#(A) = \#(B) = 2\}$. Clearly, $R(\Re) = \Delta(X) \cup \{(x,y),(y,z),(x,z)\}$. It is easy to observe that $\Re$ is not justifiable : $(\{y,z\},\{x\}) \in \Re$, although $G(X,R(\Re)) \cap \{y,z\} = \phi$. However, $\Re$ is reflexive since $\Delta([X]) \subset \Re$, and monotonic since $B \subset\subset A$ implies $\#(A) > \#(B)$,which leads to $(A,B) \in \Re$.Let $A,B \in [X]x[X]$ with $A \neq B$. If $A = X$, then $(A,B) \in \Re$. Hence suppose neither A nor B is equal to X.If $\#(A) > \#(B)$, then $(A,B) \in \Re$.Hence suppose , $\#(A) = \#(B)$.If $\#(A) = \#(B) =1$ and $(A,B) \notin \Re$, then $(B,A) \in \Re$.If $\#(A) = \#(B) =2$ then $(A,B) \in \Re$.Thus, $\Re$ is complete. In a similar fashion it can be shown that $\Re$ is transitive.
Q.E.D.

The following proposition is easy to establish.

Proposition 4 : Let $\Re$ be a PFC. If $\Re$ is justifiable, then it satisfies Monotonicity.

A PFC $\Re$ is said to be weakly justifiable if there exists an abstract game R such that $\forall A,B \in [X]$: [$(A,B) \in \Re$] if and only if [there exists $x \in A$ so that $x \in G(\{x\} \cup B,R)$].
It is easy to see that if $\Re$ is justifiable then it is weakly justifiable (: if $G(A \cup B,R) \cap A \neq \phi$, then whenever $x \in G(A \cup B,R) \cap A$, we have $x \in G(\{x\} \cup B,R)$. Further the following is easily verified :

Proposition 5 : If a PFC $\Re$ is weakly justifiable by an abstract game R, then $R = R(\Re)$. Further, in such a situation $R(\Re)$ is acyclic.

However, the following is also true:

Proposition 6: Let $X = \{x,y,z,w\}$ consist of four distinct elements. Then there exists a PFC $\Re$, which is weakly justifiable, but not justifiable.

Proof: Let $R = \Delta(X) \cup \{(x,y),(y,z),(z,y),(z,x),(w,x),(z,w),(w,z),(w,y),(y,w)\}$.R is acyclic. Define $\Re(R)$ as follows: $\forall A,B \in [X]$: [$(A,B) \in \Re(R)$] if and only if [there exists $x \in A$ so that $x \in G(\{x\} \cup B,R)$].Clearly $\Re(R)$ is a weakly justifiable PFC. However, since $y \in G(\{y,z,w\},R)$, we have $(\{x,y\},\{z,w\}) \in \Re$, although $G(\{x,y,z,w\},R) \cap \{x,y\} = \phi$, implies that $\Re$ is not justifiable.
Q.E.D.

The above proof indicates the following result whose proof is immediate:

Proposition 7: Let R be an acyclic abstract game. Then,
(a) $\Re = \{(A,B) \in [X]x[X] / G(A \cup B,R) \cap A \neq \phi\}$ implies that $\Re$ is justifiable by R;
(b) $\Re = \{(A,B) \in [X]x[X] /$ there exists $x \in A$ such that $x \in G(\{x\} \cup B,R)\}$ implies that $\Re$ is weakly justifiable by R.

3 Justifiability of Transitive PFC's

If $\Re$ is a transitive PFC then weak justifiability implies justifiability as is easily verified. Let $\Re$ be a transitive PFC and for a non empty subset $\Im$ of [X], let $\Pi(\Im,\Re) = \{A\in\Im \;/\; \forall B\in \Im: (A,B) \in \Re\}$.
A PFC $\Re$ is said to satisfy Concordance if $\forall A,B\in[X]:[(A,B)\in \Re]$ implies $[(A,A\cup B)\in \Re]$.
A PFC $\Re$ is said to satisfy Strong Concordance if $\forall A,B\in [X]:[(A,B)\in \Re]$ implies $[(A,A\cup B)\in I(\Re)]$.

Proposition 8 : (a) Strong Concordance implies Concordance; (b) Concordance plus Monotonicity implies Strong Concordance; (c) There exists a transitive PFC which satisfies Concordance but does not satisfy Strong Concordance. Hence it does not satisfy Monotonicity. Thus, it is not justifiable.

Proof: (a) and (b) are easy. Hence let us establish (c). Let $\Re = \Delta([X]) \cup \{(\{x\},\{y\}),(\{y\},\{z\}),(\{x\},\{z\})\} \cup\{(A,B) \in [X]x[X] \;/\; \#(A) < \#(B)\}\cup \{(A,B) \in [X]x[X] \;/\; \#(A) = \#(B) = 2\}$. Clearly, $R(\Re) = \Delta(X) \cup \{(x,y),(y,z),(x,z)\}$. It is easy to observe that $\Re$ is not justifiable : $(\{y\},\{x,z\}) \in \Re$, although $G(X,R(\Re))\cap\{y\} = \phi$. However, $\Re$ is reflexive since $\Delta([X]) \subset \Re$. Let $A,B\in [X]x[X]$ with $A \neq B$.If $B = X$, then $(A,B) \in \Re$. Hence suppose neither A nor B is equal to X. If $\#(A) < \#(B)$, then $(A,B) \in \Re$.Hence suppose , $\#(A) = \#(B)$.If $\#(A) = \#(B) =1$ and $(A,B) \notin \Re$, then $(B,A) \in \Re$.If $\#(A) = \#(B) =2$ then $(A,B) \in \Re$. Thus, $\Re$ is complete. In a similar fashion it can be shown that $\Re$ is transitive. However, $\Re$ is not monotonic since $\{x\} \subset\subset \{x,y\}$ but $(\{x,y\},\{x\}) \notin \Re$.Neither does it satisfy Strong Monotonicity: $(\{x\},\{y\}) \in \Re$, but $(\{x\},\{x,y\}) \in P(\Re)$.
Q.E.D.

Proposition 9 : Let $\Re$ be a transitive PFC. Then $\Re$ satisfies Concordance if and only if the following condition holds: $\forall n\in \aleph$, and $A(1),\ldots,A(n)\in[X]$, there exits $i \in \{1,\ldots,n\}$ such that $(A(i),\cup\{A(j)/j=1,\ldots,n\})\in \Re$.

Proof : Let $\Re$ satisfy the condition and let $A,B \in [X]$ with $(A,B) \in \Re$.If $(A,A\cup B)\in \Re$, then we are done. If not, then by the above condition, we have $(B,A\cup B)\in \Re$. By transitivity,
$(A,A\cup B)\in \Re$.Thus $\Re$ satisfies Concordance.
Now suppose, $\Re$ satisfies Concordance. Let $\Im_1 = \{A(j)/j=1,\ldots,n\}$. Without loss of generality suppose $A(1) \in \Pi(\Im_1 ,\Re)$. Thus, by Concordance $(A(1), A(1)\cup A(j)) \in \Re \;\forall j \in \{2,\ldots,n\}$. Let $\Im_2 = \{A(1)\cup A(j)/j=2,\ldots,n\}$. Without loss of generality suppose $A(1)\cup A(2) \in \Pi(\Im_2 ,\Re)$. Thus, by Concordance $(A(1)\cup A(2), A(1)\cup A(2)\cup A(j)) \in \Re \;\forall j \in \{3,\ldots,n\}$.By transitivity of $\Re$, $(A(1), A(1)\cup A(2)\cup A(j)) \in \Re \;\forall j \in \{3,\ldots,n\}$. Suppose that in this manner we have arrived at $(A(1), A(1)\cup A(2)\cup\ldots\cup A(i\text{-}1)\cup A(j)) \in \Re \;\forall j \in \{i,\ldots,n\}$. Let $\Im_i = \{A(1)\cup A(2)\cup\ldots\cup A(i\text{-}1)\cup A(j)/ j=i,\ldots,n\}$. Without loss of generality suppose $A(1)\cup A(2)\cup\ldots\cup A(i\text{-}1)\cup A(i) \in \Pi(\Im_i ,\Re)$.By Concordance, $(A(1)\cup A(2)\cup\ldots\cup A(i\text{-}1)\cup A(i), A(1)\cup A(2)\cup\ldots\cup A(i\text{-}1)\cup A(i)\cup A(j)) \in \Re \;\forall j \in$

$\{i+1,\ldots,n\}$.By transitivity, of $\Re$ we get $(A(1), A(1)\cup A(2)\cup\ldots\cup A(i-1)\cup A(i)\cup A(j)) \in \Re$ $\forall j \in \{i+1,\ldots,n\}$.Proceeding thus we get, $(A(1), A(1)\cup A(2)\cup\ldots\cup A(n))\in \Re$. This proves the proposition.
Q.E.D.

A similar result is available for transitive PFC's in the case of Strong Concordance.

Proposition 10 : Let $\Re$ be a transitive PFC. Then $\Re$ satisfies Strong Concordance if and only if the following condition holds: $\forall n\in \aleph$, and $A(1),\ldots,A(n)\in [X]$, there exits i $\in \{1,\ldots,n\}$ such that (a) $A(i)\in \Pi(\{A(1),\ldots,A(n)\},\Re)$; (b) $(A(i),\cup\{A(j)/j=1,\ldots,n\})\in \Re$.

Proof : Let $\Re$ satisfy the condition and let $A,B \in [X]$ with $(A,B) \in \Re$.If $(A,A\cup B)\in I(\Re)$, then we are done. If by the above condition, we have $(B,A) \in \Re$ and $(B,A\cup B)\in I(\Re)$, then we have $(A,B) \in I(\Re)$ and consequently $(A, A\cup B)\in I(\Re)$ by the transitivity of $\Re$.Thus $\Re$ satisfies Strong Concordance.
Now suppose, $\Re$ satisfies Strong Concordance. Let $\Im_1 = \{A(j)/j=1,\ldots,n\}$. Without loss of generality suppose $A(1) \in \Pi(\Im_1 ,\Re)$. Thus, by Strong Concordance $(A(1), A(1)\cup A(j)) \in I(\Re)$ $\forall j \in \{2,\ldots,n\}$. Let $\Im_2 = \{A(1)\cup A(j)/j=2,\ldots,n\}$. Without loss of generality suppose $A(1)\cup A(2) \in \Pi(\Im_2 ,\Re)$. Thus, by Strong Concordance $(A(1)\cup A(2), A(1)\cup A(2)\cup A(j)) \in I(\Re)$ $\forall j \in \{3,\ldots,n\}$.By transitivity of $\Re$,
$(A(1), A(1)\cup A(2)\cup A(j)) \in I(\Re)$ $\forall j \in \{3,\ldots,n\}$.Suppose that in this manner we have arrived at $(A(1), A(1)\cup A(2)\cup\ldots\cup A(i-1) \cup A(j)) \in I(\Re)$ $\forall j \in \{i,\ldots,n\}$. Let $\Im_i = \{A(1)\cup A(2)\cup\ldots\cup A(i-1)\cup A(j)/j=i,\ldots,n\}$.Without loss of generality suppose $A(1)\cup A(2)\cup\ldots\cup A(i-1)\cup A(i) \in \Pi(\Im_i ,\Re)$.By Strong Concordance, $(A(1)\cup A(2)\cup\ldots\cup A(i-1)\cup A(i), A(1)\cup A(2)\cup\ldots\cup A(i-1)\cup A(i)\cup A(j))\in I(\Re)$ $\forall j \in \{i+1,\ldots,n\}$.By transitivity, of $\Re$ we get $(A(1), A(1)\cup A(2)\cup\ldots\cup A(i-1)\cup A(i)\cup A(j))\in I(\Re)$ $\forall j \in \{i+1,\ldots,n\}$.Proceeding thus we get, $(A(1), A(1)\cup A(2)\cup\ldots\cup A(n))\in I(\Re)$. This proves the proposition.
Q.E.D.

Proposition 11: Let $\Re$ be a PFC. If $\Re$ is justifiable, then it satisfies Concordance.

Proof : Let $A,B\in [X]$ and suppose $(A,B)\in \Re$. Since it is justifiable, $G(A\cup B,R(\Re)) \cap A \neq \phi$. Hence, $(A, A\cup B) \in \Re$.
Q.E.D.

The following is the main theorem of this paper:

Theorem 1: Let $\Re$ be a transitive PFC. Then $\Re$ is justifiable if and only if $\Re$ satisfies Concordance and Monotonicity.

Proof : That the justifiability of a PFC $\Re$ implies Concordance and Monotonicity has been observed in Propositions 4 and 11. Hence let us assume that $\Re$ is a transitive PFC satisfying Monotonicity and Concordance.

Suppose A,B $\in$ [X] and (A,B) $\in \Re$. Towards a contradiction suppose that G(A$\cup$B,R($\Re$)) $\cap$A =ϕ. Since R($\Re$) is transitive we thus get that $\phi \neq$ G(A$\cup$B,R($\Re$)) $\subset$ B. Further the transitivity of R($\Re$) implies that G(A,R($\Re$))$\neq\phi$. Let x $\in$ G(A$\cup$B,R($\Re$)).Thus, x $\in$ B. By Monotonicity, (B,{x}) $\in \Re$. By Concordance and Proposition 9, there exists y $\in$ A : ({y}, A) $\in \Re$. This is because A = $\cup${{z}/ z $\in$A}.Thus, (x,y) $\in$P(R($\Re$)). Hence, ({x},{y}) $\in$ P($\Re$).Thus, by transitivity of $\Re$, (B,A) $\in$ P($\Re$), contradicting (A,B) $\in \Re$.Thus, G(A$\cup$B,R($\Re$)) $\cap$A $\neq \phi$.
Now, suppose that A,B $\in$ [X] and G(A$\cup$B,R($\Re$)) $\cap$A $\neq \phi$.By completeness of $\Re$, (B,A) $\in \Re$. By Concordance and Proposition 9, there exists, y $\in$ B : ({y}, B) $\in \Re$.This is because B = $\cup${{z}/ z $\in$B}. Let x $\in$ G(A$\cup$B,R($\Re$))$\cap$A. Thus, (A, {x}) $\in \Re$ by Monotonicity . Further, ({x},{y}) $\in \Re$, since (x,y) $\in$R($\Re$). By transitivity of $\Re$, we get (A,B) $\in \Re$. Thus $\Re$ is justifiable.
Q.E.D.

Given a PFC $\Re$ and sets A,B $\in$ [X], say that A is Revealed Weakly Superior (RWS) to B if G(A, R($\Re$)) $\subset$ G(A$\cup$B,R($\Re$)).
Thus, A is RWS to B, if while choosing from the union of B to A, we are not lead to the omission of elements already chosen from A. We now state a Lemma whose obvious proof is being omitted.

Lemma 1: Let $\Re$ be a transitive PFC and let A,B $\in$[X]. Then A is RWS to B if and only if G(A$\cup$B,R($\Re$)) $\cap$A $\neq \phi$.

A PFC $\Re$ is said to satisfy the Weak Revealed Preference Property if $\forall$A,B$\in$[X]: (A,B)$\in \Re$ if and only if A is RWS to B.
In view of Lemma 1, the following theorem stands established.

Theorem 2 : Let $\Re$ be a transitive PFC. Then $\Re$ is justifiable if and only if $\Re$ satisfies Weak Revealed Preference Property.

4 Justifiability of Quasi Transitive PFC's

A binary relation R on X is said to be quasi transitive if $\forall$x,y,z $\in$ X : [(x,y),(y,z)$\in$ P(R)] implies [(x,z)$\in$P(R)].
A binary relation $\Re$ on [X] is said to be quasi transitive if $\forall$A,B,C $\in$[X] :
[(A,B),(B,C)$\in$ P($\Re$)] implies [(A,C)$\in$ P($\Re$)].
It follows easily from the respective definitions that if a PFC $\Re$ is quasi transitive then so is R($\Re$) i.e. the abstract game revealed by $\Re$. We are now interested in postulating necessary and sufficient conditions for a quasi transitive PFC to be justifiable.
A PFC $\Re$ is said to satisfy Strict Concatenation if $\forall$A(1),A(2),B(1),B(2)$\in$[X]:
[(A(i),B(i))$\in$ P($\Re$), for i = 1,2] implies [(A(1) $\cup$A(2), B(1) $\cup$B(2))$\in$ P($\Re$)].

Proposition 12 : A PFC $\Re$ satisfies Strict Concatenation if and only if the following is true: for all $n \in \aleph$ and $A(i),B(i) \in [X]$ for $i = 1,\ldots,n$: $[(A(i),B(i)) \in P(\Re)$, for $i = 1,\ldots,n]$ implies $[(A(1) \cup\ldots\cup A(n), B(1)\cup\ldots \cup B(n)) \in P(\Re)]$.

The obvious proof of the above proposition is being omitted.
The following proposition is worth recording.

Proposition 13 : Let $\Re$ be a justifiable PFC. Then $\Re$ satisfies Strict Concatenation.

Proof : Let $\Re$ be a justifiable PFC and let $A(1),A(2),B(1),B(2) \in [X]$ with $(A(i),B(i)) \in P(\Re)$, for $i = 1,2]$. Hence, for $i = 1,2$ we have $G(A(i)\cup B(i),R(\Re))\cap B(i) = \phi$. Thus, whenever $x \in B(1) \cup B(2)$, there exists $y \in A(1) \cup A(2)\cup B(1) \cup B(2)$, such that $(y,x) \in P(R(\Re))$.Thus, $G(A(1)\cup A(2)\cup B(1)\cup B(2),R(\Re))\cap[B(1)\cup B(2)]=\phi$. Since, $\Re$ is justifiable, it is clearly the case that $[(A(1) \cup A(2), B(1) \cup B(2)) \in P(\Re)]$. Thus, $\Re$ satisfies Strict Concatenation.
Q.E.D.
A PFC $\Re$ is said to satisfy Strict Monotonicity if $\forall A,B,C,D \in [X]$: $[A \subset B \subset C \subset D$ and $(C,B) \in P(\Re)]$ implies $[(C,A) \in P(\Re)$ and $(D,B) \in P(\Re)]$.

Proposition 14 : Let $\Re$ be a justifiable and quasi transitive PFC. Then $\Re$ satisfies Strict Monotonicity.

Proof : Let $\Re$ be a justifiable PFC and let $A,B,C,D \in [X]$ with $[A \subset B \subset C \subset D$ and $(C,B) \in P(\Re)]$.Hence, $G(B\cup C,R(\Re))\cap B = \phi$. Thus, whenever $x \in B$, there exists $y \in B\cup C$, such that $(y,x) \in P(R(\Re))$.Since $C \subset D$, whenever $x \in B$, there exists $y \in B\cup D$, such that $(y,x) \in P(R(\Re))$.Hence, $G(B\cup D,R(\Re))\cap B = \phi$ .Thus, $(D,B) \in P(\Re)$.
Now, $G(B\cup C,R(\Re))\cap B = \phi$ and $R(\Re)$ is quasi transitive implies that whenever $x \in B$, there exists $y \in G(B\cup C,R(\Re)) \subset C$, such that $(y,x) \in P(R(\Re))$.Since $A \subset B$, whenever $x \in A$, there exists $y \in C$, such that $(y,x) \in P(R(\Re))$. Hence, $G(A\cup C,R(\Re))\cap A = \phi$. Thus, $(C,A) \in P(\Re)$.
Q.E.D.
Note: In the proof of the above theorem the quasi transitivity of $\Re$ was used only to establish that part of Strict Monotoniciy which says that $\forall A,B,C \in [X]$: $[A \subset B \subset C$ and $(C,B) \in P(\Re)]$ implies $[(C,A) \in P(\Re)]$.It is not required to establish the part of Strict Monotonicity which says that $\forall B,C,D \in [X]$: $[B \subset C \subset D$ and $(C,B) \in P(\Re)]$ implies $[(D,B) \in P(\Re)]$.This latter part holds for any PFC which is justifiable.
A PFC $\Re$ is said to satisfy Weak Expansion if $\forall A,B \in [X]$ and $x \in X$: $[(\{x\},A) \in \Re$ and $(\{x\},B) \in \Re]$ implies $[(\{x\},A\cup B) \in \Re]$.

Proposition 15 : Let $\Re$ be a justifiable PFC. Then $\Re$ satisfies Weak Expansion.

Proof : Suppose $\Re$ is justifiable,$A,B \in [X]$,$x \in X$ and $[(\{x\},A) \in \Re$ and $(\{x\},B) \in \Re]$. Since $\Re$ is justifiable , $x \in G(A\cup\{x\},R(\Re))\cap G(A\cup\{x\},R(\Re))$.Thus, $\forall y \in A\cup B$: (x,y)

$\in R(\Re)$.Thus, $x \in G(A\cup B\cup\{x\},R(\Re))$. The justifiability of $\Re$ implies that $(\{x\},A\cup B) \in \Re$, thus proving the proposition.
Q.E.D.
A PFC $\Re$ is said to satisfy Weak Condorcet if $\forall A\in [X]$: $[x \in G(A,R(\Re))]$ implies $[(\{x\},A) \in \Re]$.

Proposition 16 : Let $\Re$ be a PFC satisfying Weak Expansion. Then $\Re$ satisfies Weak Condorcet.

Proof: Let $\Re$ be a PFC satisfying Weak Expansion and suppose that for some $A\in[X]$ it is the case that $x \in G(A,R(\Re))$.Hence, $\forall y\in A$: $(x,y) \in R(\Re)$.Hence, $\forall y\in A$: $(\{x\},\{y\}) \in \Re$.By Weak Expansion, $(\{x\},A) \in \Re$.
Q.E.D.
We can now prove the following:

Theorem 3: Let $\Re$ be a quasi transitive PFC. Then $\Re$ is justifiable if and only if $\Re$ satisfies Strict Concatenation, Strict Monotonicity and Weak Expansion.

Proof :Let $\Re$ be a quasi transitive PFC.Then, it follows from Propositions 13,14 and 15 that $\Re$ satisfies Strict Concatenation, Strict Monotonicity and Weak Expansion. Hence suppose that $\Re$ satisfies Strict Concatenation, Strict Monotonicity and Weak Expansion. Let $(A,B) \in \Re$ and towards a contradiction suppose that $G(A\cup B,R(\Re)) \cap A = \phi$. Thus, $G(A\cup B,R(\Re))\subset B$. Since, $\Re$ and consequently $R(\Re)$ is quasi transitive, $\forall x\in A$, there exists $y(x) \in G(A\cup B,R(\Re))$, such that $(y(x),x) \in P(\Re)$. Since $\Re$ satisfies Strict Concatenation, Proposition 12 yields $(\{y(x)/x\in A\},A) \in P(\Re)$.By Strict Monotonicity and the fact that $y(x) \in G(A\cup B,R(\Re)) \subset B$ whenever $x \in A$ we get $(B,A) \in P(\Re)$. This contradicts, $(A,B) \in \Re$. Hence $G(A\cup B,R(\Re)) \cap A \neq \phi$.
Now suppose that $A,B \in [X]$ and $G(A\cup B,R(\Re)) \cap A \neq \phi$.Towards a contradiction suppose that $(A,B) \notin \Re$.By completeness of $\Re$, we have $(B,A) \in P(\Re)$. By Strict Monotonicity, $(A\cup B,A) \in P(\Re)$.Let $x \in G(A\cup B,R(\Re)) \cap A$. By Strict Monotonicity and $(A\cup B,A) \in P(\Re)$, we get $(A\cup B,\{x\}) \in P(\Re)$. However by Weak Expansion and Proposition 16, $(\{x\}, A\cup B) \in \Re$, contradicting $(A\cup B,\{x\}) \in P(\Re)$.Thus, $(A,B) \in \Re$. Thus, $\Re$ is justifiable.
Q.E.D.

Acknowledgement: I would like to thank Prasanta Pattanaik for motivation in this research.

References

M.A. Aizerman and F. Aleskerov (1995): *Theory of Choice*, North Holland.

M.A. Aizerman and A.V. Malishevski (1986): Conditions for Universal Reducibility of Two-Stage Extremization Problem To a One-Stage Problem, *Journal of Mathematical Analysis and Application*, 119, pp.361-388.

F. Aleskerov (1999): *Arrovian Aggregation Models*, Kluwer Academic Publishers.

K.J.Arrow (1994): *A Note on Freedom and Flexibility,* K.Basu, P.Pattanaik, K.Suzumura (eds.),*Choice, Welfare, and Development.A Festschrift in Honour of Amartya K.Sen,*Clarendon Press, Oxford.

M. Klemisch-Ahlert (1993): Freedom of Choice: A Comparison of Different Rankings of Opportunity Sets, *Social Choice Welfare 10*, 189-207.

T.C. Koopmans (1964): On the Flexibility of Future Preferences, in M.W. Shelley and G.L.Bryan (eds.), *Human Judgments and Optimality*, Wiley, New York.

D. M. Kreps (1979): A representation theorem for 'Preference for flexibility', *Econometrica* 47,565-577.

D.M. Kreps (1988): Notes on the Theory of Choice, *Underground Classics in Economics,* Westview Press, Boulder, London.

S. Lahiri (2000): Reducing a multi-stage vector optimization problem to a single stage vector optimization problem ,*Opsearch* 37(2),124-133.

S. Lahiri (2001): The Lexicographic Composition of Abstract Games, (forthcoming in) *Economic Theory.*

A.V. Malishevsky (1997): An axiomatic justification of Scalar optimization, in Andranik Tangian and Josef Gruber (eds.): Constructing Scalar valued objective Functions, *Lecture Notes in Economics and Mathematical Systems*, Springer-Verlag.

P.K.Pattanaik and Y.Xu (1990): On ranking Opportunity Sets in Terms of Freedom of Choice , *Rech.Econ.Louvain* 56(3-4):383-390.

P.K.Pattanaik and Y.Xu (1997): On Diversity and Freedom of Choice, *mimeo.*

P.K.Pattanaik and Y.Xu (1998): On Preference and Freedom, *Theory and Decision*, 44, 173-198.

C.Puppe (1995): Freedom of Choice and Rational Decisions, *Social Choice Welfare 12*, 137-153.

C.Puppe (1996): An Axiomatic Approach to 'Preference for Freedom of Choice', *J.Econ.Theory* 68:174-199.

A.K. Sen (1988): Freedom of Choice : Concept and Content, *Europ. Econ. Rev*. 32, 269-294.

A.K. Sen (1990): Welfare, Freedom and Social Choice: A Reply, *Rech. Econ. Louvain,* 56, 451-485.

A.K. Sen (1991): Welfare ,Preference and Freedom, *Journal of Econometrics* 50, 15-29.