Predatory equilibria: Systematic theft and its effects on output, inequality and long-run growth[*]

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We present a model in which agents can devote energies to production or to appropriating the fruits of other people's labour. We investigate the situations under which such transfers are equilibria, i.e. will reproduce themselves over time. We note that many of worst outcomes can be observed when players are relatively evenly matched and when the social environment makes predatory activity very successful. In these situations we may even see economic collapse. The introduction of property rights has many of the expected effects, with rewards to productive activity increasing. Nevertheless these gains materialise only if the protection is stronger in areas where the more productive player has a comparative advantage. It is possible to achieve more cooperative outcomes in the repeated game, but paradoxically this might lead to higher levels of income transfers.

[*]The current form of the paper owes an enormous debt to Anne Case, who kept prodding me to clarify what my abstract model had to say about real issues. If I have not succeeded better in this, it is certainly not her fault. Comments from Stefan Schirmer and Johannes Fedderke improved the product markedly.

Explaining the wealth and poverty of nations and individuals has been one of the abiding concerns of political economy. The existence or absence of property rights has often been seen as a crucial variable in explaining the di¤erent trajectories of di¤erent societies. Landes (1998), for example, has suggested that the reason why Europe grew faster than China, was that the fragmented nature of political power imposed limits on the ability of the ruling classes to extract resources from the producers.

The concern with the impact of predatory behaviour on the well-being of society goes back at least to Adam Smith. Indeed, Mittermaier (1999) has suggested that Adam Smith's famous "invisible hand" argument should in fact be interpreted as an argument about the deleterious e¤ects of predation: If everyone were only to concentrate on producing the maximum that they were capable of, then, of necessity, the aggregate product would be as large as it could possibly be. It is when people concentrate their energies not on production, but on the appropriation of other people's output that the overall product falls short of the maximum possible.

Predation involves two costs: …rstly there is the production foregone as a result of the energy devoted to it; secondly there is the cost incurred by producers in protecting themselves against it. Indeed, parasitism may be viewed as a negative sum game: the total product is lower when one agent is parasitic than when everyone cooperates.

This raises an interesting set of issues. If predation removes the fruits of honest labour would it not lower the incentives for honest people to produce? If so, is there not the possibility of landing in a downward spiral in which eventually no one produces? Can predation lead to the collapse of societies, or would it be possible to have an "equilibrium" level of parasitism? And if so, is this equilibrium unique or is it possible to have more or less parasitic types of equilibria?

The historical evidence suggests that situations in which a class of people systematically extract resources from another group can persist for very long times. Landes's (1998) account of China is a case in point. He suggests that the imperial Chinese model was based on the link between "numbers, food and power" (1998, p. 23). Lots of cultivators were mobilised on potentially arable land, to grow crops that in turn supported the armies and the centralised administration. Centralised control in turn allowed the deployment of yet more cultivators on yet more arable land and so on. Productivity improvements in this system (such as irrigation schemes) ultimately bene…tted the imperial centre more than the cultivators.

The Landes account thus suggests that it is not only the existence of a predatory equilibrium which needs explanation, but also how the equilibrium shifts in relationship to technological and social change. Conversely the nature of the equilibrium (i.e. how exploitative it is) will presumably in‡uence the possibility of growth and accumulation.

In this paper we will be concerned to model situations in which there is a long-term equilibrium in which one party ends up continously transferring resources to another one. We will try to explain the levels of the transfers as well as the level of overall production in such societies. We will assess how changes in productivity, in the technology of appropriation and in wealth a¤ect the outcome of the interaction. We will ask the following kinds of questions:

- How does inequality a¤ect the type of transfers that occur? The Landes account of Chinese development suggests that there are situations in which it is the rich and powerful who steal from the poor. Under what circumstances might we expect this kind of behaviour and when is it the poor who steal from the rich? What is the impact of inequality on the output of the society?

- What is the impact of productivity change? We might assume that the more productive a person is, the greater the cost to that person of engaging in parasitic activities; also, the more tempting it would be for the less productive members of that society to sponge o¤ her. Who, therefore, ends up bene…tting from productivity improvements?

- What happens if the technology of appropriation changes? The development of certain types of technologies (guns, centralised bureaucracies, deeds registries) might make it easier or more di¢cult to appropriate the output. One would assume that changes that make claiming activity more decisive would lead to a shift towards non-productive activities.

- What is the impact of property rights? Landes's central thesis is that property rights are crucial to the long-run performance of economies. How might we model this? What other impacts might property rights have on the way the economy functions?

- Under what circumstances might we expect predation to be minimised? Is the Smithian idea of a society in which everyone expends maximal

> energy on production utopian, or are there institutional arrangements that would bring it about?

The model through which we will investigate these and other questions is a variant of the “parasitism” model (Wittenberg 1999). Our starting point is a two-player interaction in which each player has a choice about how much energy to expend on production and how much on the appropriation of the product that has already been produced. The optimal level of production versus claiming behaviour will be dependent also on the decisions of the opponent and it is the nature of this strategic interaction which drives the analysis. We allow players to di¤er both in how productive they are, as well as in how e¤ective they are in making claims.

We consider initially the short-run, one-period interactions and the nature of the equilibria that might be attained. We then consider how these di¤er from the long-run equilibria. It transpires that in the long run the level of inequality become endogenous to the model. The payo¤s attained in each period by each player determine the long-run allocation of wealth. Consequently our model can be thought of not only as a model of equilibrium transfer of resources, but also of equilibrium inequality.

The structure of the discussion will be as follows. In Section 1 we present the simplest version of the one-period interaction. In this model we do not, as yet introduce property rights. We show how changes in inequality, productivity and claiming strength a¤ect the choice between production and predation. In Section 2 we endogenise the level of inequality (by considering the long-run equilibrium). We interpret this model as being perhaps the closest in spirit to the peasant-feudal lord interaction. We consider a few other archetypal patterns of interaction. In Section 3 we show what happens when the decisiveness of claiming increases to relatively high levels. It transpires that it is then possible to get multiple equilibria. Furthermore small shifts in the control parameters may then lead to catastrophic jumps in the equilibrium levels of inequality and production.

Section 4 shows that there are a number of di¤erent ways in which one might think about the costs of predation, e.g. direct transfers, investment in claiming activities and opportunity costs. These do not necessarily highlight the same features of the situation. We show that predation seems to be most costly when the players are relatively evenly matched. In Section 5 we extend the model to include multi-player interactions. We show that to some extent the behaviour of the model does, indeed, conform to the stylised picture presented by Landes of the Chinese imperial state.

In Section 6 we develop a more elaborate model in which we now have property rights. We proceed to investigate how the resultant equilibria differ from those discussed earlier. We note that in some circumstances even a small amount of predation can have enormous consequences, for the level of inequality and production in society. In most sensible cases it turns out, however, that property rights improve the conditions of the more productive player. This has important benefits for the level of output and growth rates of the economy.

In Section 7 we consider whether there are any prospects of achieving Pareto superior outcomes to the Nash equilibria considered throughout this paper. Given the repeated nature of the interactions in our model, this ought to be possible. The major points to consider, however, are that stages of our game build on each other and do not simply repeat the same set of options. This means that each player cannot afford to adopt strategies which would effectively remove her from the interaction.

We conclude by reflecting back on some of the issues raised earlier. Proofs and more detailed analytical discussion are provided in the appendix.

Throughout the discussion we assume that the players interact without the mediation of an external agency or state, i.e. our agents exist in a Hobbesian "state of nature" or in Hirshleifer's (1995) state of "anarchy". This assumption is not only to simplify the analysis, but also in view of the fact that the state can frequently itself be seen as a party to the interaction, or as an agent of one of the players.

1 Description of the basic model

Hirshleifer's (1995) anarchy model in fact serves as the starting point for our analysis. In this model he imagines that there is a fixed resource (e.g. a gold field or rich agricultural area) which is available for utilisation. However there is no external agency which allocates claims: the agents have to stake out and fight for their claim, as well as productively exploit their portion of the overall resource. Each agent is, as it were, involved in a Hobbesian fight of all against all. The problem her is how to split the available resources between productive effort and fighting effort so as to maximise overall income. Devoting more resources now to fighting effort will lead to greater resources, which in the future will lead to greater income. On the other hand, increasing fighting effort means that current production is at a lower level.

One of the limitations of this model is that the underlying resource base is not affected by the conflict. Although there are costs to engaging in conflict in terms of production foregone, the resource base itself cannot be run down or destroyed.

In order to deal with this problem we have modified the Hirshleifer model. In our model the conflict is not about a pre-given resource - it is about the aggregate product of that society itself. As in the anarchy model, agents face a choice: they can either contribute to production or they can expend energy on staking claims to the output. If the agents do not spend a sufficient proportion of resources on production, however, they will end up fighting over a shrinking pie.

Formally our model can be described as follows: there are two players A and B with initial endowments of wealth W_A and W_B respectively. The choice confronting each player is what proportion of these resources to devote to productive and to claiming activities respectively. We label the proportions devoted to production as α and β respectively. We assume that the aggregate production function is given by Y_{AB} which has the following characteristics: $\frac{\partial Y_{AB}}{\partial \alpha} > 0$, $\frac{\partial Y_{AB}}{\partial \beta} > 0$ and $Y_{AB}(0;0) = 0$. Indeed we will make the more restrictive assumption that the production function is separable in the products of A and B, i.e. $Y_{AB} = Y_A + Y_B$, where Y_A and Y_B are the outputs of A and B respectively. With this specification it is possible to unambiguously identify the contributions of A and B, which helps to identify whether the final allocation exhibits parasitism or not. In the discussions below we have chosen particular functional forms given by:

$$Y_A = c_A(\alpha W_A)^h, \; Y_B = c_B(\beta W_B)^h \tag{1}$$

where $0 \leq \alpha; \beta \leq 1$; c_A and c_B are productivity parameters and h is a return to scale parameter. While the model can be developed for the more general case (see Wittenberg 1999, for a discussion of the increasing returns to scale case) we will restrict our discussion here to the situation where $h = 1$.

The appropriation functions g_A and g_B determine how the aggregate output is split up between the players. We assume that g_A and g_B depend on the respective energy that is put into claiming by A and B. Furthermore we assume that g_A and g_B are homogeneous of degree zero in $(1 - \alpha)$ and $(1 - \beta)$. This implies that the division of the product depends only on the relative resources devoted to appropriation. In particular it implies that the actual contributions made to the total product are not a consideration when the aggregate product is divided up.

As a consequence, a player that does not manage to stake any claim, will receive nothing, even if that player has contributed the largest share of the total output. Claiming activities should therefore not be thought of as intrinsically illegitimate. People who do not engage in e¤orts to establish their rights to particular resources or then to defend those rights are likely to be taken advantage of. In this sense claiming is not only an alternative to production, but also a necessary complement (if the other player is likely to engage in claiming, that is).

This view of human nature is not that far fetched. Entire professions have grown up around the establishment and enforcement of claims. Litigation, the registration of title deeds or the registration of patents are all examples of claiming activities in this sense. Lobbying government for welfare payments or for a reduction of taxes would be others. Furthermore the outcomes of these contestations need not be related to the intrinsic merits of the cases, but may often just re‡ect the relative skills of the lawyers or politicians involved.

We assume that g_A and g_B are given by

$$g_A = \frac{s_1^m}{s_1^m + s_2^m} \text{ and } g_B = \frac{s_2^m}{s_1^m + s_2^m} \tag{2}$$

where s_1 and s_2 represent the relative strengths of players A and B in making claims on the output. We assume that s_1 is a function of $(1-\alpha)$ but not of $(1-\beta)$ while the reverse is true of s_2, with $\frac{\partial s_1}{\partial(1-\alpha)} \geq 0$ and $\frac{\partial s_2}{\partial(1-\beta)} \geq 0$.

The parameter m is a decisiveness parameter - it records how sensitive the …nal division of aggregate output is to claiming behaviour. With a low m claiming activities are relatively ine¤ective and the …nal output is more or less equally divided. With high m claims become highly e¤ective and the …nal shares come to re‡ect the respective energy that was put into making claims on the output. With an extremely large m, the person with the largest muscle gets to keep everything.

It should be noted that m is a re‡ection of the social values and technologies available within a society. We might list some of them as follows:

- cultural factors: A society's attitude towards wealth and inequality would de…nitely a¤ect m. A great belief in equality would tend to reduce m, while a high tolerance for inequality would drive up m:

- military technology: The more sophisticated the tools of destruction, the more leverage the owners of those implements would tend to have on the division of the product, i.e. this would drive up m.

- storage technology: Limits on the ability to store and transport wealth (e.g. the presence or absence of grain silos) would tend to reduce m.

Again we have chosen particular functional forms for the claiming strength functions, given by

$$s_1 = (1 - \alpha) f_A W_A$$
$$s_2 = (1 - \beta) f_B W_B$$

Note that $(1 - \alpha) W_A$ and $(1 - \beta) W_B$ are the respective resources devoted to claiming activities. The parameters f_A and f_B can be seen as conversion parameters - they determine how resources (wealth) get translated into claiming strength. They indicate that different agents may not have access to the same fighting or claiming technology. In the case of feudal societies knowledge about how to produce certain types of weapons would have been closely guarded secrets of the court. In more recent times, apartheid South Africa imposed legal restrictions on the ability of black South Africans to get access to guns. There were also restrictions on the kinds of legal claims that blacks could make on property. All of these would have severely impaired the efficiency with which claims could be established

The final payoffs to each player are given by

$$Y_1 = g_A [c_A \alpha W_A + c_B \beta W_B]$$
$$Y_2 = g_B [c_A \alpha W_A + c_B \beta W_B]$$

Our key concern is to analyse the effects of the strategic interactions around appropriation and production. It is evident that if both players claim only and do not produce, then there will be no product to split. This, however, cannot be an equilibrium: if the other player is determined to be an absolute parasite, it would be in my interest to produce something, because even a small share of a positive output would be preferable to absolute no return at all. The balance between appropriation and production that we will see ought to depend on the productiveness of the players, their effectiveness in establishing claims, their respective wealth and the degree to which claiming is an effective activity.

We will in general be more concerned with analysing the effects of relative changes in wealth and productivity. We therefore reparameterise our model, letting

$$k = \frac{W_A}{W_B}, \quad p = \frac{c_A}{c_B} \quad \text{and} \quad f = \frac{f_A}{f_B} \tag{4}$$

We interpret k as our index of inequality, p as an index of productivity differentials and f as an index of A's relative claiming strength. Without loss of generality we will assume throughout that B is the less productive individual, i.e. $p \geq 1$. With this reparameterisation, c_B and W_B now function as scale parameters. Increases in c_B and W_B (for fixed values of p and k) lead to increases in the productivity or wealth of both players. To signal this change, we drop the subscript[1].

We can therefore write the payoffs as

$$Y_1 = \frac{(1-\alpha)^m f^m k^m}{(1-\alpha)^m f^m k^m + (1-\beta)^m} c\,(p\alpha k + \beta)\,W \tag{5a}$$

$$Y_2 = \frac{(1-\beta)^m}{(1-\alpha)^m f^m k^m + (1-\beta)^m} c\,(p\alpha k + \beta)\,W \tag{5b}$$

We will make the Cournot assumption that players treat their opponent's choice of cooperativeness as fixed. This means that the one-period equilibrium will be at the intersection of the respective reaction functions, where these give the optimal values of α (or β) given the opponents choice of β (or α). For interior solutions the reaction functions will be given by the loci of the solutions to

$$\frac{\partial Y_1}{\partial \alpha} = 0 \quad \text{and} \quad \frac{\partial Y_2}{\partial \beta} = 0$$

Figure 1, however, indicates that we are not guaranteed to get interior solutions. There will be combinations of the parameters for which one of the players becomes completely parasitic. Indeed the possibility of complete parasitism and its effects on the interactions with the other player turn out to be absolutely crucial for the behaviour of the model.

[1] A similar point applies to f, of course. The "baseline" appropriation efficiency f_B does not feature in the payoff function, however, so there is no need to concern ourselves with this issue.

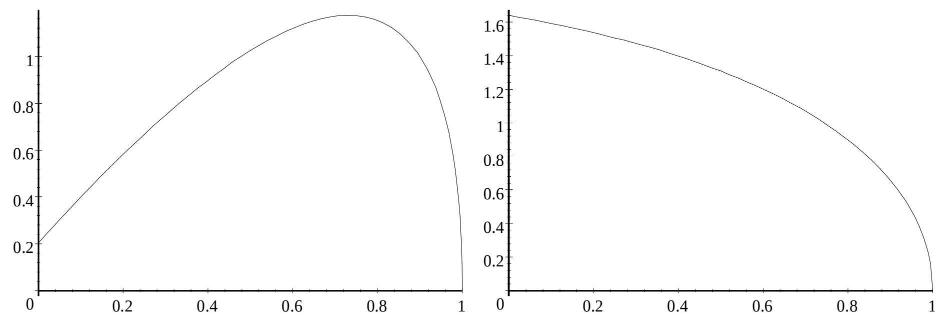

Figure 1: Plot of Y_2 against choice of β. For some choices of the parameters, the payoff function has its global maximum in the interior of $[0;1)$, for others the payoff function is monotonically decreasing. Case (a): $k = 1$, $p = 1$, $f = 1$, $W = 1$, $c = 4$, $m = 0{:}5$, $\alpha = 0{:}1$. Case (b)$k = 10$, $p = 1$, $f = 1$, $W = 1$, $c = 2$, $m = 0{:}5$, $\alpha = 0{:}6$.

In Figure 2 we have graphed some reaction functions and the locus of Cournot equilibria for a particular set of parameters. It is evident that beyond a critical level of inequality ($k = 5$ in the diagram), Player B becomes completely parasitic. Indeed for higher levels of inequality B's reaction function moves very sharply down to zero and then stays at zero until A shows excessively high degrees of cooperation.

We detect three kinds of reaction functions in the diagram:

1. One type (exemplified by A's reaction function when $k = 10$) which increases monotonically over the interval $[0;1)$

2. Another (exemplified by both A's and B's reaction functions when $k = 1$ or $k = 2$) which first decreases and then increases, but always has an interior solution to the optimisation problem $\frac{\partial Y_1}{\partial \alpha} = 0$ or $\frac{\partial Y_2}{\partial \beta} = 0$.

3. A final kind (exemplified by B's reaction function when $k = 10$) which monotonically decreases to zero, then coincides with the line $\beta = 0$ and then increases monotonically towards $\beta = 1$.

We can show (in Theorem 13) that these are indeed the only cases to be found. Furthermore it is not accidental that a type 1 reaction function on the part of A is paired with a type 3 reaction function of player B. This

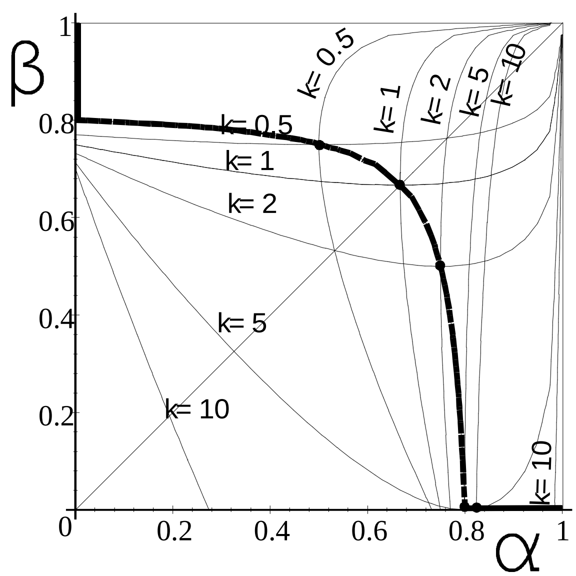

Figure 2: Reaction functions and Cournot equilibria change with the inequality parameter k. The locus of equilibria is indicated by the heavy line. Large dots indicate intersections of the reaction functions in the diagram. Parameters: f = 1, p = 1, m = 0:5.

will always be the case: whenever one player's reaction function reaches the boundary, then the other player's reaction function will be monotonically increasing (see corollary 14).

Some additional properties of the reaction functions are evident from the diagram. In each case the reaction functions start at a high level. Indeed the optimal response to $\alpha = 0$ for B involves a choice for β such that $\beta \geq \frac{1}{m+1}$. This is due to the fact that if A is completely parasitic B is compelled to produce, if B wants a positive payoff. As α increases, this compulsion gradually disappears. In our diagram B's response is to immediately increase her levels of claiming (i.e. B's reaction functions are of type 2 or 3). This is due to two factors. At higher levels of α A is producing positive output which adds to aggregate output. This increases the incentive for B to claim. On the other hand, as α increases, A's level of claiming activity goes down. This makes B's claims relatively much more effective. The combination of increased incentive with increased effectiveness leads to the decrease in the reaction functions observed in the diagram. This decrease happens until either B becomes a complete parasite or until the Cournot equilibrium is reached. In the case of complete parasitism the reaction function eventually reappears and converges on $\beta = 1$ at a high enough level of α. Essentially at these levels of α A is doing such little claiming that B is left with almost the total output. In these situations it is in B's interests to start producing to increase the total output. Although B becomes more cooperative, the reaction function stays far below the 45^o line in this region, so B devotes relatively much larger resources to claiming than A.

A's optimal choice is obviously influenced by similar considerations. One additional point to note is that if A is substantially wealthier than B (e.g. $k = 10$ in the diagram), then we have a type 1 reaction function: there is no longer any incentive for A to increase her claiming activities as B becomes more cooperative. The additional resources that she can obtain from her own production far outweigh the benefits she might gain by claiming from B.

Note that there are two types of Cournot equilibria. In the first kind both players produce a positive output and the reaction functions intersect at their respective minima. The second kind involves one player becoming completely parasitic. The reaction function of the non-parasitic player in this case increases. In both cases therefore, the Cournot equilibrium has the curious property that it represents the maximally uncooperative point on either player's reaction function. Equivalently, it is the point at which both players spend the most energy on claiming, given that this level must be rational on some hypothesis about the opponent's behaviour.

The exact equilibrium that is reached is dependent on the parameters of the model. The diagram provides graphical evidence that increasing k would make A more cooperative and B less so. It is less evident how the other parameters would affect the outcome. The comparative statics for this single period interaction can be summarised as follows (see Theorem 29):

1. Inequality of wealth: An increase in A's wealth relative to B makes A more cooperative and B less so:

   $$\frac{\partial\alpha}{\partial k} \geq 0, \ \frac{\partial\beta}{\partial k} \leq 0$$

   The reason for this is evident: as a player becomes more wealthy, a smaller proportion of resources devoted to claiming will have the same effect. Consequently the individual who becomes more affluent can afford to devote more resources to production.

2. Relative productivity: If A becomes more productive relative to B, then A would tend to become more productive:

   $$\frac{\partial\alpha}{\partial p} \geq 0, \ \frac{\partial\beta}{\partial p} \leq 0$$

   The result is again intuitive: the costs of production foregone increase for the person who becomes more productive. For the less productive individual the gains from own production start looking less attractive relative to what can be gained by claiming from the other player.

3. Changes in claiming efficiency: If A becomes more effective in establishing claims relative to B, then A becomes more productive:

   $$\frac{\partial\alpha}{\partial f} \geq 0, \ \frac{\partial\beta}{\partial f} \leq 0$$

   This result makes sense if one remembers that an increasing claiming effectiveness implies that A gets to keep a larger share of her output. It therefore becomes in her interest to enlarge the output. For the player who loses ground in the claiming stakes, it is in their interest to increase their claiming effort and so reduce the productive effort.

4. Changes in the decisiveness of claiming: If claiming becomes more decisive, then we would expect both players to spend more time claiming and less time producing. In fact the result is more complicated

$$\frac{\partial \alpha}{\partial m} \lesseqgtr 0 \text{ as } \frac{g_A}{g_B} \lesseqgtr \gamma, \quad \frac{\partial \beta}{\partial m} \lesseqgtr 0 \text{ as } \frac{g_B}{g_A} \lesseqgtr \gamma$$

where $\gamma \approx 3.591121$. If the players are relatively evenly matched, then an increase in the decisiveness parameter would unambiguously increase the amount of claiming activities. If there is a large imbalance in the power of the players, then the player who currently extracts the lion's share would actually become more productive, since this power now obviously translates into a much larger impact.

2 Long-run parasitism: feudal exploitation

The results thus far are for one period interactions only. We will be concerned to investigate Cournot equilibria that might persist for many periods - decades or even centuries. We are therefore interested in equilibria which are intertemporally stable, i.e. the relative wealth k at the end of the period must be the same as at the beginning of the interaction. In order to do this, we need to close the model somehow. The simplest way to do this is to abstract away from consumption and simply convert the entire current period returns into next period wealth, i.e.

$$W_{A;t+1} = Y_{1;t} \qquad W_{B;t+1} = Y_{2;t} \tag{6}$$

Consequently $k_{t+1} = \frac{Y_{1;t}}{Y_{2;t}}$ which we can write (substituting in equations 5a and 5b) as

$$k_{t+1} = \frac{(1 - \alpha_t)^m f^m}{(1 - \beta_t)^m} k_t^m \tag{7}$$

We have subscripted α and β with t to indicate that in general α and β would not need to be constant from period to period. Of course in an equilibrium they would be. These equilibrium values of α and β will depend on the parameters k, p, f and m. It is also evident from this equation that if $m > 1$, the dynamics of the model are likely to be somewhat unstable.

Provided that $m \leq 1$, there is, in fact, a unique value of k which yields a stationary state. We can therefore think of equation 7 as determining k as a function of m, p and f, i.e.

$$k = k(m; p; f)$$

Indeed it can be shown (see theorem 30) that for an interior Cournot equilibrium we will have

$$k = \left(\frac{f}{p}\right)^{\frac{m}{m+1}} \tag{8}$$

This is a remarkable equation, both for its simplicity and for what it implies. It suggests that the relative wealth of any player is positively related to the player's own claiming efficiency and inversely related to own productivity.

We might interpret this scenario as the Seven Samurai scenario: the more productive individual (read: the peasant) produces every period at a high level only for the richer and militarily stronger parasite (read: the bandit, or the feudal lord) to expropriate a large share of the output (the harvest) and thus reproduce the inequality in wealth and power from period to period. What makes this equilibrium stable is the inability of the poorer more productive individual to squirrel resources away to take on the parasite. What is perhaps even more startling is that increases in the productivity of the peasant increase the disparity in wealth between the peasant and the lord, thus benefitting the lord. Equation 8 suggests that the feudal lord would be richer, the more productive his peasant is and the stronger his own military capability.

Nevertheless, the story is somewhat more complicated than this. In Figure 3 we have graphed the equilibrium level of k for changing values of p. On the graph we have also indicated the boundary lines within which there is an interior solution, i.e. within which the equilibrium value of k is given by equation 8. At $p \approx 504$ this solution is no longer valid and we reach a corner solution. At this point the equilibrium k no longer depends on p, indeed the value will be given by the solution to the equation

$$f = k^{\frac{1-m}{m}}\left(\frac{m+k+1}{m}\right) \tag{9}$$

Increases in peasant productivity therefore lead to changes in the wealth of the lord up to a point determined by the military strength of the lord

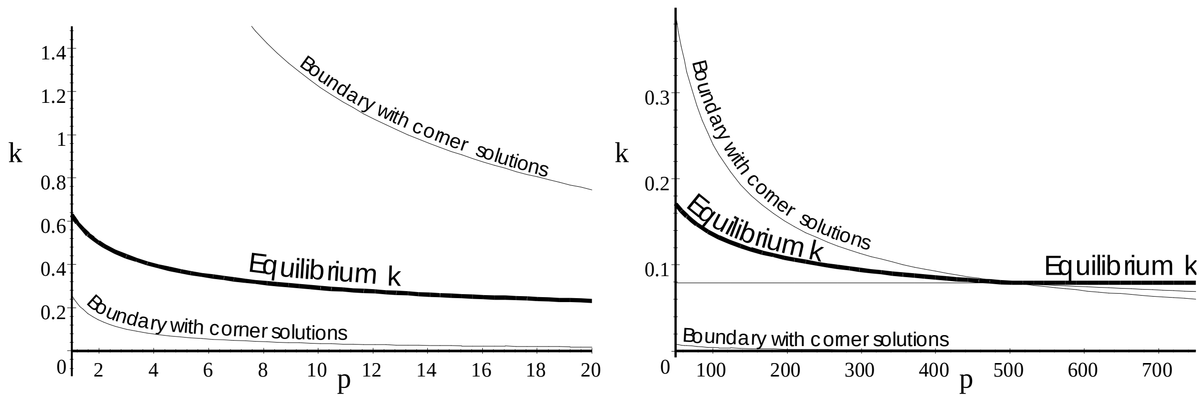

Figure 3: The lord gets richer as the peasant gets more productive, until p ' 504. Then inequality remains stable. Choice of parameters (for both panels): m = 0:5, f = 0:25. Left panel: 1 · p · 20. Right panel: 50 · p · 750.

(given by f) and the extent to which the prevailing social conditions favour appropriation (as re‡ected by m).

The movements along the equilibrium k curve therefore re‡ect the changing balance of forces as the richer, less productive player concentrates progressively on feeding o¤ the poorer more productive one. Once the richer player has become a complete parasite, however, he can no longer do any more damage, as it were. From this point on the bene…ts of increases in productivity will be shared proportionately (which of course implies that the parasite gets the lion's share).

This raises a question about how the poorer player responds to the predation. There are two e¤ects that need to be taken into consideration. On the one hand, increases in p lead to an incentive e¤ect. As the peasant becomes more productive, there are greater incentives to produce (as we noted above). On the other hand, we have also seen that in equilibrium much of this extra production is siphoned o¤ by the lord, increasing wealth disparities. This wealth e¤ect would tend to o¤set the incentive e¤ect. In Figure 4 we show that the incentive e¤ect initially predominates, but beyond a certain point (when p ' 152) the wealth e¤ect leads the peasant to reduce output. At the point where the other player becomes completely parasitic, the wealth e¤ect ceases to matter and ® remains constant. The level of this equilibrium, as before, depends only on f and m.

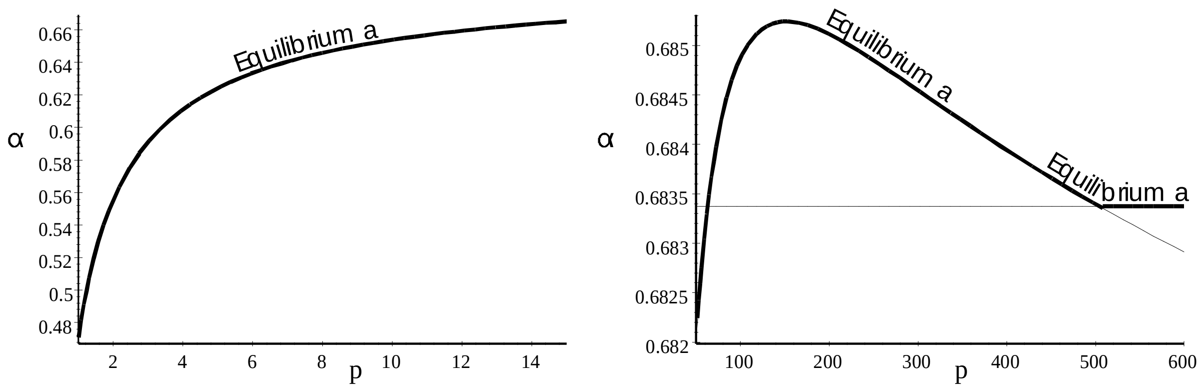

Figure 4: Increases in p have two e¤ects: …rstly they increase the incentive for A to produce, but they also increase the disparity in wealth. The incentive e¤ect initially outweighs the wealth e¤ect, but the peasant eventually cuts down somewhat on production. At the point at which player B becomes a complete parasite, the optimal ® no longer depends on p. Choice of parameters: m = 0:5, f = 0:25. Left panel: 1 · p · 15. Right panel: 50 · p · 600.

The relative productivity of the peasant (p) is only one of the factors helping to determine the level of inequality in equilibrium. As noted above, the relative e¤ectiveness of the lord in appropriation (as measured by f) is also important. Decreases in f (i.e. increasing e¤ectiveness in appropriation by the lord) will (by equation 8) lead to a decrease in the equilibrium level of k, i.e. an increase in the relative wealth of the lord. A decrease in f will therefore have an unambiguous e¤ect on the productive e¤ort by the peasant: it will reduce it directly and also through the concomitant wealth e¤ect. This raises the prospect that if the richer player's advantage in the contest for resources becomes too great, the poorer player will cease production altogether and become a complete parasite. This is show in Figure 5. At the point at which this happens, the equilibrium level of k is no longer given by equation 8. Instead it is now given by the solution to the equation

$$f = \frac{k^{\frac{1}{m}} m}{km + k + 1} \tag{10}$$

As shown in the diagram, at this point further increases in B's strength would lead to even greater inequality. Up to this point, decreases in f can be

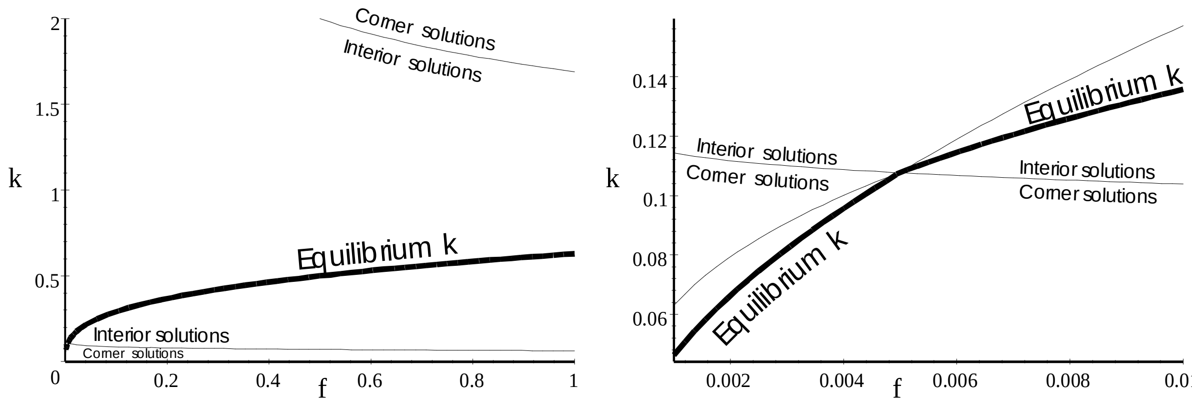

Figure 5: As player B's claiming e¤ectiveness increases (i.e. f decreases), k decreases, i.e. B gets richer. At a critical level (f ' 0:004987) player A becomes completely parasitic. At this point inequality increases at a more rapid rate. Choice of parameters: p = 4, m = 0:5.

o¤set by player A devoting more energy to claiming. Once A has abandoned all productive activities, however, this strategy is no longer open, and so any additional strength by B translates directly into additional wealth for B.

Of course with such a turn of events the relationship between the poorer player and the richer one could not realistically be described as feudal[2]. The poorer (though more productive) player would be a pauper - relying on begging, theft or robbery to make a living. Would there be precedents for such situations? Another example from the "Wild West" genre of movies is the con‡ict between the wealthy rancher and the small scale homesteaders. Another would be the enclosure movement. The ability of the rich landowners to lay claim to the communal land areas squeezed the poorer peasants to the point where it made some sense to become full time beggars or highwaymen. Many peasants would, of course, also have left the area to look for opportunities in the cities. Exit from the game is, of course, not an option in our model.

We have therefore isolated two polar cases:

[2] Landes makes the distinction between "feudal" relationships, which are between a lord and his vassal, i.e. they are intra-nobility relationships and "manorial" or "seigneurial" relationships which are those between the lord and his serfs. We have used the term "feudal" in its more colloquial sense to cover the latter relationships.

	$p < 1$	$p > 1$
$f < 1$	Cottager/Vagrant/Bandit May be richer or poorer. Will never be exploited	Peasant Will always be poorer May be exploited (feudal case) May be parasitic (landless peasant)
$f > 1$	Lord/Landowner Will always be richer May be parasitic (feudal case) May be exploited (landowner)	Farmer May be richer or poorer Will never be parasite

Table 1: Equilibrium outcomes of the two-player game for player A

- One where the rich parasite appropriates from the poor productive individual (the feudal case) and
- One where the poor parasite lives off the scraps from the rich producer (the landless peasant vs the landowner).

Between these outcomes there are many intermediate ones, in which both players produce some output. To the extent to which the richer player is the one producing less, the dynamics will have more of a feudal flavour; where the richer player is also more productive it will resemble the landowner/landless peasant interaction more.

In the examples above we have assumed throughout that player A is both more productive and less strong than player B. It is of course possible for these two variables to combine in other ways. The possible combinations (from player A's perspective) are shown in Table 1. We have labelled the cases in ways that might be suggestive of empirical situations. Player B's role (for given values of f and p) will be given by the diagonally opposite cell in the table, i.e peasant is always paired with lord/landowner and farmer with cottager.

Clearly the actual outcome reached will depend not only on f and p, but also on m. Equation 8 suggests that an increase in m will have the effect of widening inequality. If $k > 1$, it will increase, while if $k < 1$, it will decrease. It turns out that this is true even if we have reached a corner. In all cases an increase in decisiveness will benefit the player who is richer at the expense of the player who is poorer. The response of the poorer player is to reduce the level of productive activity (where this is positive).

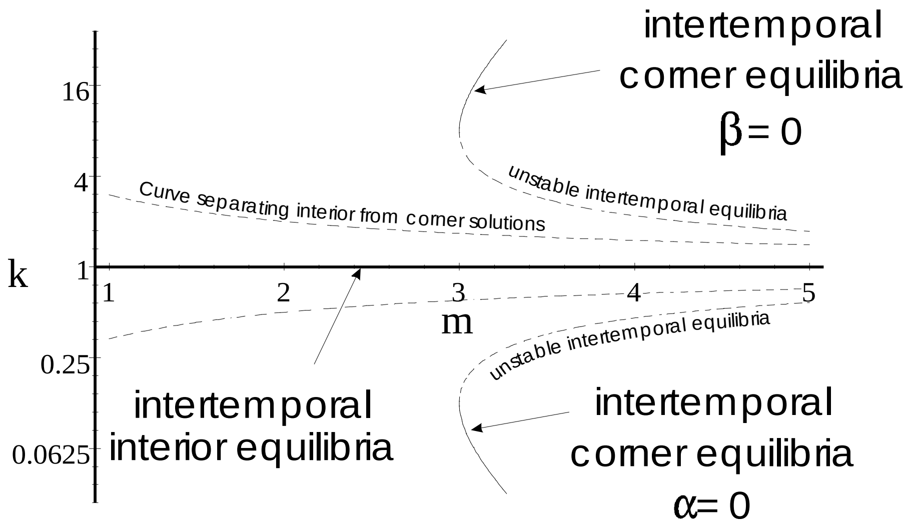

Figure 6: If m > 1, there may be multiple equilibria for the same choice of p; f and m. Choice of parameters: f = p = 1

3 The dark ages: turmoil and trouble

What happens if decisiveness increases such that m > 1? Clearly many of the broad conclusions reached above still hold. Nevertheless in this new environment, several additional outcomes are possible.

The …rst, and perhaps most dramatic, change is that it is now possible to get multiple equilibria corresponding to the same levels of f, p and m. In Figure 6 we have graphed the curves representing intertemporal equilibria for the special case f = p = 1. In this perfectly symmetrical case, the symmetrical distribution of wealth k = 1 is obviously a stable solution. Nevertheless at su¢ciently high levels of m it turns out that there are also equilibria with unequal distributions of wealth. In the diagram there are therefore no fewer than …ve equilibria for any m > 3, three stable ones and two unstable ones. Clearly the equilibrium that is reached will depend on the initial conditions. In the region where there is only one intertemporal equilibrium, all starting values of k will eventually converge on the equilibrium value. In the zone where there are multiple equilibria, the loci of unstable equilibria serves to separate out the initial conditions which will converge on the symmetric equilibrium from those which will converge on the corner equilibria.

The second new possibility is that of “catastrophic” changes in the equilibrium level of inequality (and hence activity levels) consequent on small changes in the control parameters. This possibility is already present in the previous example. If initial conditions have led to one of the corner equilibria, then smooth decreses in m will eventually lead to a discontinuous jump in k. In the example, this would happen at the latest at $m = 3$. Adjustments towards a lower m would lead to a jump in k from either 8 or 0.125 to 1. This would be the situation in which baronial privilege would suddenly collapse consequent on a decrease in society’s tolerance of inequality. In this example, however, it is not clear that increases in m would ever shift the equilibrium away from $k = 1$.

In Figure 7 we indicate that jumps may be pervasive if $m > 1$. In this case the productivity parameter p and the claiming e¤ectiveness parameter f just balance each other so that while there is an interior equilibrium, wealth remains equally divided. Once the corner is reached, however (at $m = 2$) a small increase in decisiveness would lead to a marked shift in favour of the producer. This example might be reminiscent of the increase in farmer wealth consequent on the expulsion of the cottagers from the commons. With very high levels of m there is another set of equilibria which become intertemporally stable, associated with the less productive player being richer and being the sole producer.

What happens to the level of production along these di¤erent curves? We have noted above that the poorer individual would tend to restrict output consequent on an increase in m. This introduces yet another possibility. We may see a reversal in who is the producer and who the parasite. This is shown in Figure 8 where initially we are dealing with a feudal lord-peasant interaction. As society’s tolerance for inequality increases, a point is reached where the peasant has cut down production to such an extent, that it becomes necessary for the lord to produce. Increases in m will lead to reduction in peasant output, to the point where the peasant has …nally ceased producing. One might wish to think of this point as the point where the lord has successfully converted all the land to ranch land. What is, of course, interesting about this is that the same individual attributes (represented by f and p) in di¤erent social environments, (represented by m) will lead to qualitatively quite di¤erent outcomes.

In this …gure we note yet again that there is a possibility of getting multiple equilibria for su¢ ciently large m. Focusing on the continuous adjustment curve in the bottom half, we note that there are three changes of slopes evi-

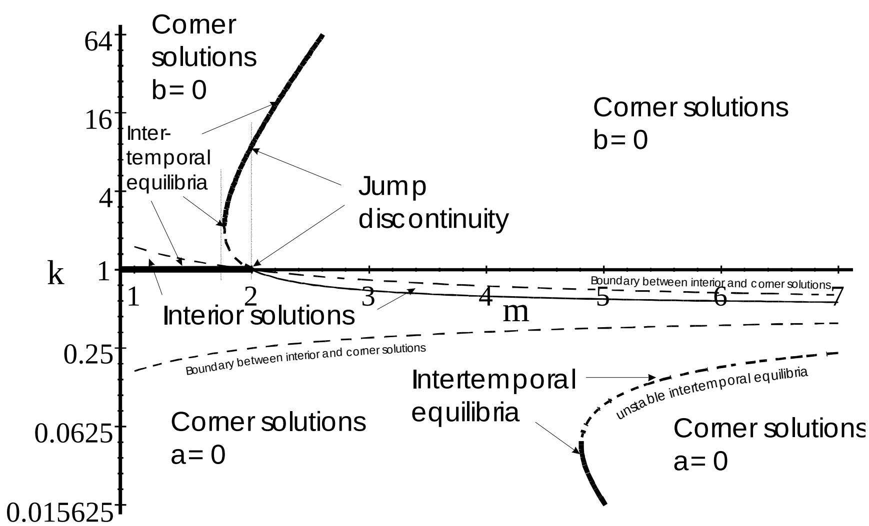

Figure 7: For small m the only stable equilibrium is at k = 1. Somewhere between m ' 1:76364 and m = 2 an increase/decrease in m will be accompanied by a jump in the equilibrium k. At m = 2 and k = 1, for example, an increases in m would lead to a jump to the new equilibrium k = 9. Choice of parameters: f = p = 2.

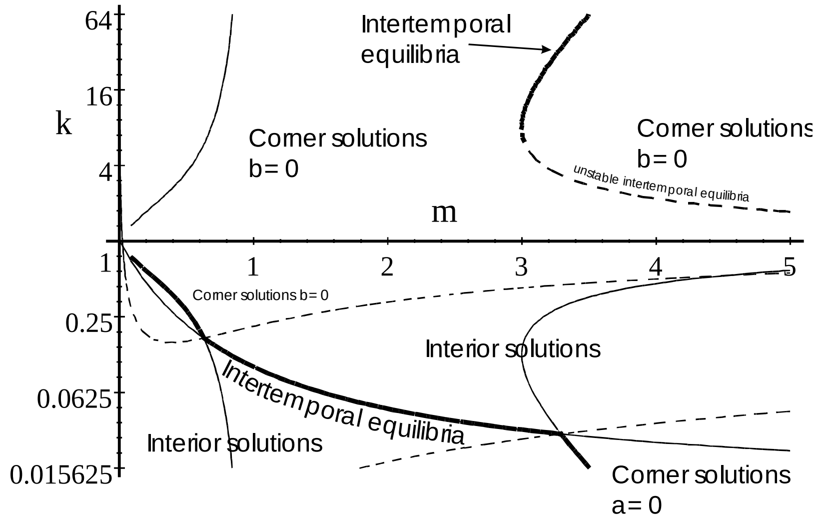

Figure 8: As m increases, player A changes from being the sole producer to being a complete parasite. Player B makes the opposite transition. Choice of parameters: f = 1, p = 100.

dent, representing the transitions into, out of and into parasitism respectively. The …rst transition is barely evident, very close to m = 1. At the second kink, i.e. the point where player B enters production, inequality increases at a slower rate than before. This is due to the fact that with B producing a positive output, claiming activity by A is no longer geared only to protecting her own output, but increasingly towards appropriating part of B's production. Once A has become completely parasitic, however, she cannot do any more damage. Consequently this check on the rate of growth of inequality is removed.

We have seen that the bottom path in the main represents movements by A from sole producer to complete parasite and the converse movement on the part of B. What happens in the equilibria at the top right hand corner? Since these are all corner equilibria, B remains a complete parasite at all points along the curve. In the case of A, however, there are two distinct trajectories. The "upper" trajectory is again the stable one. It is characterised by the fact that k is increasing, while k is decreasing along the unstable one. It turns out that ® will also be increasing along the former and decreasing along the latter trajectory.

Indeed in general there may be two types of corner equilibria, the outer (stable) one in which inequality is most extreme, which will be associated with increasing levels of production by the sole producer and the inner (unstable) one which is marked by decreasing levels of production. For large enough m the stable trajectories will always exist. Indeed we can show that $k > f^m(m-1)^m$ if B is the complete parasite and $k < \frac{f^m}{(m-1)^m}$ if A is the parasite. This shows that inequality increases exponentially with m along these trajectories. We might dub these trajectories aggressive equilibria because they represent situations in which the marginal player is getting completely squeezed out of the picture.

What is less clear is how aggregate output changes along these trajectories. We have graphed both output[3] and the growth rate in Figure 9 to correspond to the situation depicted in Figure 8. There are a number of really stark conclusions. In the first place output falls along the continuous adjustment path. This makes sense, since the more productive individual is gradually reducing her productive effort. Interestingly, however, the output and the growth rate bounce back a little, once player A has become completely parasitic. The reason is that increases in decisiveness benefit B (who is the richer), since they allow B to reduce the effects of A's predations. In this model the worst of all possible outcomes is to reach the point at which the more productive player has just been marginalised. It is much better[4] to completely annihilate the influence of the landless peasant, than to let her hang on at the margins. The outcomes attainable along the continuous adjustment path are, however, decidely inferior to the outcomes attained (at a high enough m) with the more productive individual as the sole producer. The outcomes with the highest output and growth rates all involve the complete destruction of the influence of the less productive individual, i.e. they are "aggressive equilibria".

The actual outcomes that are achieved depend on the levels of c and W. The latter influences the level of output, whereas the former affects both the output levels and the growth rates. In the bottom right panel of Figure 9 we show that it is possible to get negative growth rates. Interestingly enough in the diagram these occur only around the point where player A just becomes parasitic. Along the "aggressive" trajectories we find that in the limit the

[3] It should be noted that the output graphs are somewhat misleading, since the aggregate wealth which generates the output is not constant as m changes. This defect does not apply to the growth rate graphs.

[4] for output and growth rates, though obviously not for A

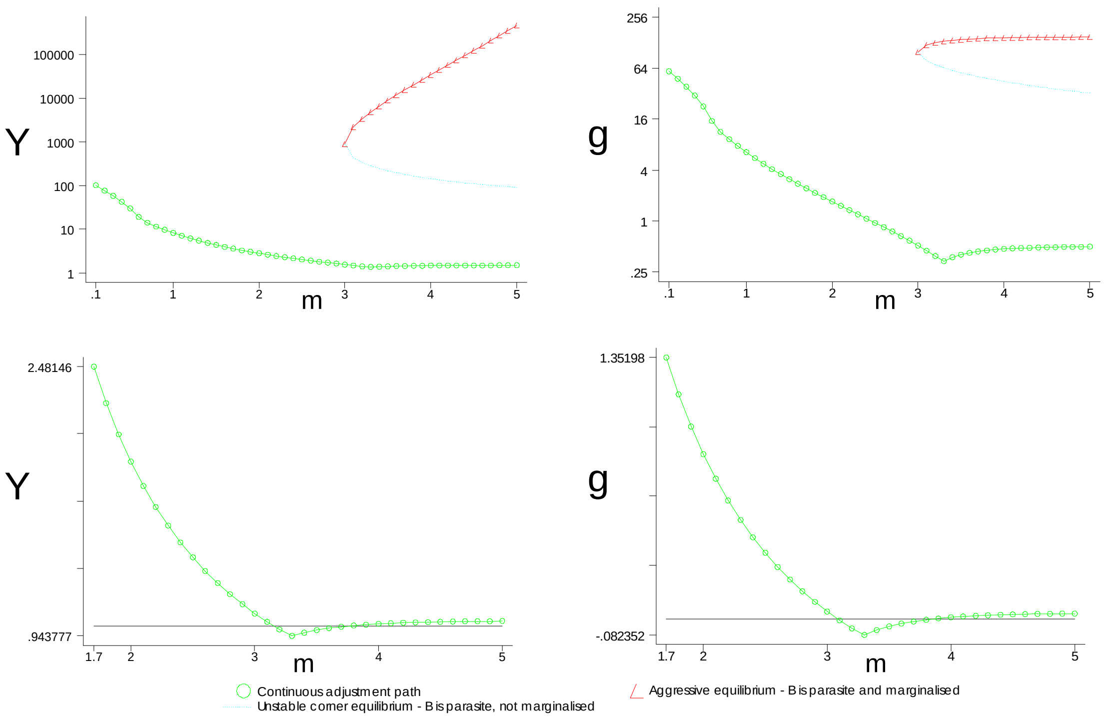

Figure 9: Output and growth rates decrease with m unless one player is completely parasitic, when they might increase. There are some equilibria which are much more productive than others. The baseline productivity c a¤ects both the level of output and the growth rate. Indeed in some cases the growth rate may be negative. Choice of parameters: $f = 1$, $p = 100$, $W = 1$. Top row: $c = 1{:}5$, bottom row: $c = 1{:}03$. Bottom row depicts only outcomes near $Y = 1$ and $g = 0$.

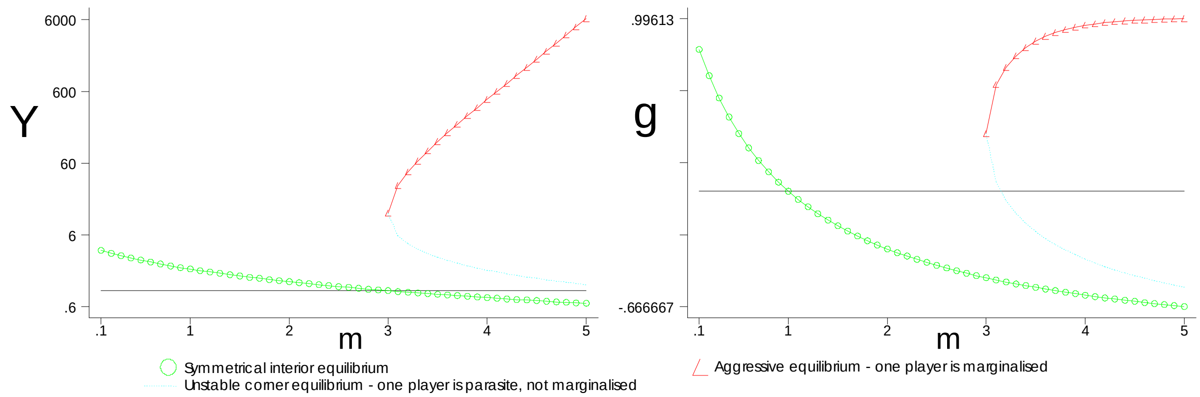

Figure 10: When players are evenly matched, an increase in m will eventually lead to negative growth rates. Choice of parameters: c = 2, f = 1, p = 1, W = 1.

growth rates converge to c_A ¡ 1 and c_B ¡ 1 respectively. Positive growth is therefore possible only if the player who is left as the sole producer is su¢ciently productive.

The possibility of wholesale collapse is, however, not restricted to cases in which the productivity parameter is barely above one. In Figure 10 we have graphed the situation of completely evenly matched players (depicted also in Figure 6). We chose the productivity parameter c = 2, so that if each player produced to maximum capacity, wealth would double each period. As is evident, however, with increasing decisiveness this society achieves negative growth rates for all m > 1. The only exception to this is along the aggressive corner equilibria. The lesson again seems clear: in a context where claims become highly decisive the worst outcome is where the contenders are evenly matched, so that maximum energy gets exerted on claiming activities.

The idea that society could move from stable expansion to economic contraction and collapse with relatively modest movements in the decisiveness parameter should not be surprising to students of economic history. The conventional history of the “dark ages” pictures the situation in more or less this fashion: with bands of marauding raiders on the loose, productive activity was cut back. Only where a local strong leader was able to annihilate such parasitic claimants, would growth have been possible. Nevertheless such “local” solutions would have been vulnerable to the sorts of catastrophic collapse noted earlier.

The combination of multiple equilibria, catastrophic jumps and destructive competition among evenly matched protagonists would in more or less any situation merit the description of the "dark ages".

4 Land clearances and the cost of parasitism

Several ways of thinking about the prevalence of conflict and parasitism have been alluded to in the discussion thus far. The most obvious one is simply the effort exerted by each player in production (i.e. the level of α and β) versus the effort devoted to claiming (i.e. $1 - \alpha$ and $1 - \beta$). The last few examples, however, have indicated that the opportunity costs are also an important measure. Indeed there are at least four ways that we might think about measuring parasitism:

1. The energy devoted to claiming: In our model this would be indexed by

$$I_{\alpha} = 1 - \alpha \text{ and } I_{\beta} = 1 - \beta \tag{11}$$

   In an empirical setting one might want to estimate this by the number of criminals, beggars and so on. One limitation with this index is that α and β are bounded below by zero and once this bound has been reached this index is silent about what happens within society.

2. The level of transfers from A to B as a proportion of output: In our model this index would be calculated as

$$I_T = \frac{Y_A - Y_1}{Y_A + Y_B} \tag{12}$$

   where $Y_1 = g_A[Y_A + Y_B]$. Negative values of this index are of course transfers from B to A. This index tracks the gap between what A produces and what A finally gets as a proportion of total output. In an empirical setting this might be proxied by the total loss due to criminal activity. One disadvantage with this index is that there may be no or relatively few net transfers in the armed camp scenario: where everyone is armed to the teeth, but all this claiming balances out so that in the end everyone keeps more or less what they have produced.

3. The proportion of resources devoted to claiming: This can be represented as

$$I_C = \frac{(1-\alpha)W_A + (1-\beta)W_B}{W_A + W_B} \tag{13}$$

Empirically this would be the sum of investments made in security arrangements (fortifications, fences, alarms and security guards) and in criminal equipment (cannons, battering rams, guns, lock-picking devices). It will be useful to note that this index is in fact a weighted average of the claiming intensities I_α and I_β, weighted by the shares of aggregate wealth.

4. The opportunity cost of claiming activities: Obviously if everyone was producing and no one was claiming aggregate output would be higher. The loss in potential output due to claiming (per unit of initial wealth) is

$$I_O = \frac{c_A(1-\alpha)W_A + c_B(1-\beta)W_B}{W_A + W_B} \tag{14}$$

Another way of thinking about this index is that it represents the difference between the maximally attainable growth rate g_{max} and the actual growth rate g. Empirically this is probably the hardest to estimate although it would perhaps also be the most interesting. One disadvantage with this particular index is that it is not bounded above. We will therefore often work with a rescaled version of this index:

$$I_L = \frac{c_A(1-\alpha)W_A + c_B(1-\beta)W_B}{c_A W_A + c_B W_B} \tag{15}$$

This index, like I_C, is also a weighted average of the claiming intensities, but weighted by the contribution to maximum potential output. Perhaps a somewhat more intuitive way of thinking about it is as the difference between the maximal growth ratio[5] π_{max} and the actual

[5] where $\pi_{max} = \frac{c_A W_A + c_B W_B}{W_A + W_B}$, i.e. it is the "gross" growth rate. Obviously $\pi_{max} - 1 = g_{max}$ and $\pi - 1 = g$. Note that $\pi_{max} - \pi = g_{max} - g$. The numerator of I_L is therefore the difference in the growth rates.

growth ratio ¼, when expressed as a proportion of the maximal growth, i.e.

$$I_L = \frac{¼_{max} \;\; i \;\; ¼}{¼_{max}}$$

In Figure 11 we have graphed four of the indices, $I_®$, I_T, I_C and I_L for the completely symmetrical case. We note that the symmetrical (and interior) solution k = 1 is associated with no net transfers of resources. Nevertheless on every one of the other indices it suggests that there is a higher level of parasitism. This is the “armed camp” scenario suggested earlier. Large quantities of resources are devoted to claiming, but in the resulting stalemate everyone manages to keep just what they produce.

By contrast with this, the “aggressive” equilibrium in which B becomes marginalised ends up showing the lowest level of parasitism on virtually all indicators: A produces at a high level, there are only small amounts of resources transferred, only a small proportion of all resources is devoted to claiming activities and production approaches the maximally possible level. As suggested earlier, if claiming becomes decisive, the worst outcome occurs when the protagonists are evenly matched.

A much more complicated picture is presented in Figure 12, which presents $I_®$, I_T, I_C and I_L for the case corresponding to Figures 8 and 9. In all of the indices the transition out of and into parasitism (at m ' 0:6283604682 and m ' 3:291456415 respectively for the continuous adjustment path) are marked. The quality of these transitions is very di¤erent, however. Both I_C and I_T record a decrease in parasitism once B starts producing. In the case of I_C, however, the decrease in parasitism is very pronounced. The reason for this is lies in the fact, alluded to earlier that I_C is a weighted average of $I_®$ and $I_–$, with the weights given by the shares of wealth. I_C is therefore extremely sensitive to decreases in parasitism by the wealthier player. I_T by contrast, depends on the relative productivity of the players as well. The levels of transfers in this case remain high, despite the fact that the wealthier player is spending fewer resources on claiming. It is interesting to note that player A remains exploited almost right up to the point at which she herself becomes completely parasitic.

I_L, by contrast to I_C and I_T marks a decline in parasitism only once A has become completely parasitic. The reason for this is that with the massive imbalance in productivity in the case under review, a decrease in

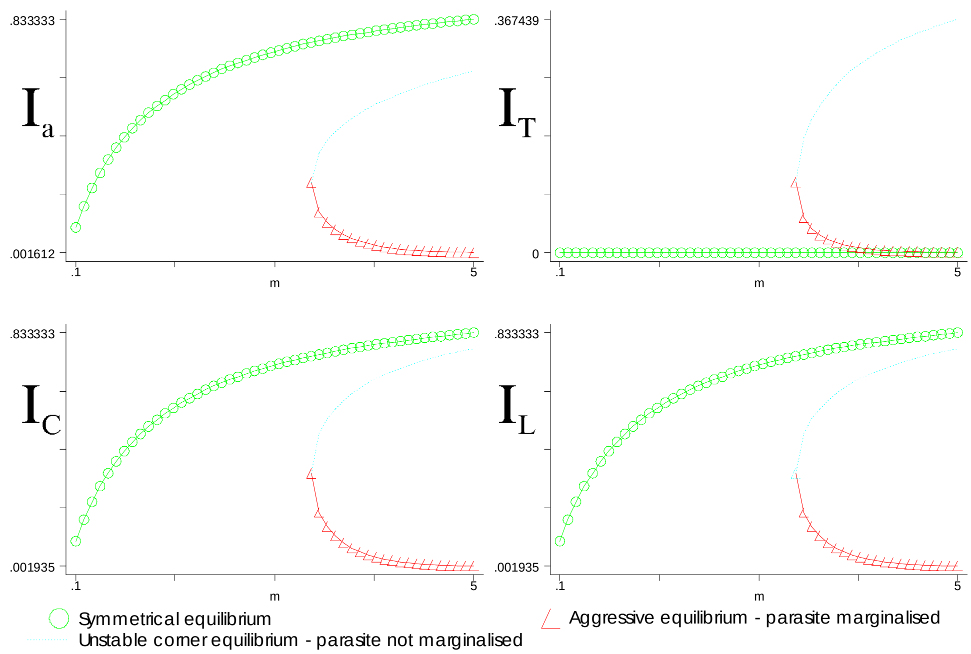

Figure 11: Di¤erent indices of parasitism in the "armed camp" scenario. When the players are evenly matched, no net resources are transferred (as shown by I_T), but large amounts of energy are absorbed in claiming. Choice of parameters: $f = p = 1$, $c = 2$, $W = 1$.

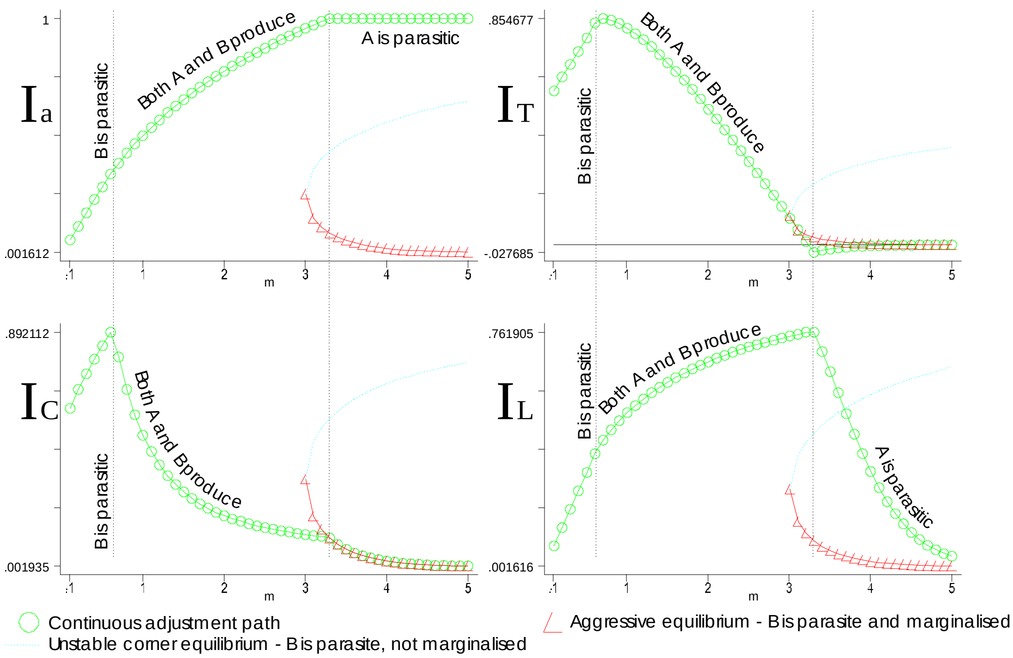

Figure 12: Di¤erent indices of parasitism highlight di¤erent features. All indices show the importance of the transitions when one or the other agent becomes completely parasitic. These are indicated for the continuous adjustment path by the dotted lines. Choice of parameters: f = 1, p = 100, c = 1:5, W = 1.

A's level of production has a much larger e¤ect on the potential output than the corresponding increase by B. This e¤ect becomes less pronounced as the resources with which A could produce become reduced (as shown by the decrease in k in Figure 8). As A becomes marginalised, the contribution she could make to aggregate production with the resources at her disposal shrink to zero.

This interplay between what is possible and what is actual is shown in more detail in Figure 13, where, for clarity, we have presented the same information twice: once in ordinary levels (left hand panel) and once on a log scale (right panel). The gap between actual and potential output widens as A moves into parasitism. In this range A still manages to control some resources, but they need increasingly to be put towards claiming activities, to ward of the predations of B. It is this gap between what A could be

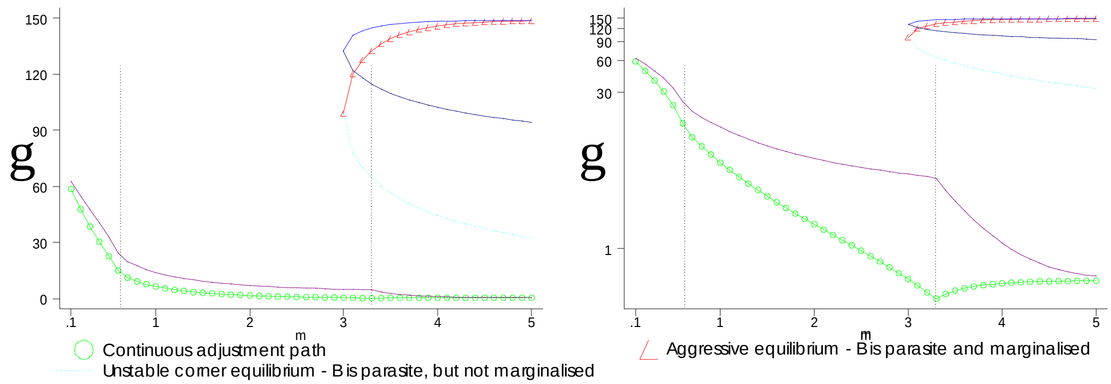

Figure 13: Actual and potential growth rates. Up to the point where A becomes completely parasitic, the decrease in ® is faster than the decrease in k, leading to a much larger wedge between actual and potential growth. Choice of parameters: f = 1, p = 100, c = 1:5, W = 1. Left panel: growth rates. Right panel: growth rates on a log scale.

producing and what she is actually producing that is recorded as an increase in I_L in diagram 12. Once A's resource base has been eroded (k decreases very fast once A is completely parasitic), her potential contribution dwindles to zero. Furthermore in this range B becomes able to devote more resources to production. Hence the gap between actual and potential growth narrows and so I_L shrinks to zero.

Despite the fact that I_L approaches zero in both aggressive equilibria, these are clearly not equivalent. In the one there is a growth rate approaching 0:5, while in the other it is approaching 149. The "best" outcome is achieved in the aggressive equilibrium where the more productive player (A) manages to completely marginalise the less productive one (B). This raises an important point about the way in which the gap between actual and potential growth in diagram 13 ought to be interpreted. The curve depicting the maximum possible growth is for a given level of k, which in turn depends on the parameters f, p, and m. Maximal growth is therefore relative to a particular income distribution which itself depends on the interplay of social forces. The counterfactual considers only what would happen if there was a mutual cessation of claiming with the given level of k. If, however, we were also to consider redistributions of income, the maximal growth rates would, in fact increase. Indeed if p > 1 the maximal growth would always be obtained if the most productive player was given all the resources to produce with. This is, of course, what happens in the limit in the aggressive equilibrium where B is marginalised.

The conclusions from all these cases are clear:

- There may be large costs incurred in claiming even when there are no net transfers occurring
- There may be large transfers of resources even when the resources devoted to claiming seem to be falling
- Even when both the levels of transfers and the resources devoted to claims are falling, the opportunity cost of claiming may be very high
- Conflict and all the costs that are associated with it are minimised if one of the players is marginalised; growth and output are maximised if the player who is removed is also the less productive one!

The last point has some resonances with the different trajectories of Argentina and the United States, as related by Landes (1998, Chapter 20). In the former it was the less productive ranchers who ended up dominating, while in the latter it was the homesteaders and the squatters who prevailed over the rich landowners (see especially pages 318-320).

The marginalisation of one of the players is possible only in a context of a very high m, i.e. the society must have a high tolerance for inequality, the technology of appropriation must be highly decisive and the output of the society must be such that it can be easily appropriated. These combinations are probably found most readily in the context of conquest and colonisation. Indeed, one might think of the different trajectories of different types of colonialisms in terms of both the relative productivity of the conqueror versus the conquered and the decisiveness with which the conquest happened: Land clearances and wholesale dispossession of the indigenous populations in the United States, Canada and Australia can be contrasted with the extractive relationship developed with the original inhabitants in other parts of the world.

5 "Numbers, food and power": The rise and fall of chiefs and empires

The differences between land-grabbing conquests (the marginalisation of a competitor) and the extraction of tribute has probably existed for as long as people have lived in organised societies. The pursuit of tribute, however, raises a number of interesting issues in the context of the current model. Existing wealth can be used not only to make claims of an existing serf, but to enslave additional producers. How might we extend the model to deal with many players?

The simplest way would be to divide players into two types (peasants and lords) and to permit a lord to have many dyadic interactions. Each interaction could then be modelled as a two-player game, exactly as before. In order to make the analysis tractable we would need to assume that if the lord interacts with n peasants, he would be able to bring only one n-th of his resources to bear on each individual interaction, i.e. he would not be able to concentrate his resources - first on browbeating one recalcitrant peasant and then another. If we assume that all peasants are identical (i.e. we have the same p and f in every game), then the outcome of every interaction would be exactly as analysed before, except that the disparity between the wealth of any one peasant and the total wealth of the lord would now be given by $k = \frac{k^*}{n}$ where k^* is the equilibrium level of inequality from the corresponding two-player game. Inequality would therefore widen. In the specific context of interior intertemporal equilibria we would have

$$k = \frac{1}{n}\left(\frac{f}{p}\right)^{\frac{m}{m+1}}$$

The assumption that the n interactions are independent of each other is somewhat unpalatable, since it implies that the peasants cannot see that additional resources are going into the societal pot, so while they might be able to contest how the outcome of their labour is split with the Lord, they cannot influence how any of the other divisions are made. This isolation of individual interactions from each other might, however, in some circumstances be justified. Marx once famously compared the French peasants to a sack of potatoes:

> The small-holding peasants form a vast mass, the members of which live in similar conditions but without entering into manifold relations with one another. Their mode of production isolates

> them from one another instead of bringing them into mutual intercourse. .... A small holding, a peasant, and his family; alongside them another small holding, another peasant and another family. A few score of these make up a village, and a few score of villages make up a Department. In this way, the great mass of the French nation is formed by simple addition of homologous magnitudes, much as potatoes in a sack form a sack of potatoes. (Marx 1977, p.317)

Of course if the interactions are independent of each other, then the outcomes of the individual interactions are the same if there is one lord facing n peasants or n (identical) lords facing n peasants. Indeed there may be all sorts of intermediate outcomes: one lord with n_1 peasants and another with n_2 where $n = n_1 + n_2$, and so on. The peasant "base" could therefore support all kinds of feudal superstructures. This is, of course, more or less exactly how feudalism did work. Fiefs expanded and contracted as noble families fought or married one another. Peasants kept their heads down and paid their dues to whoever happened to be in the local manor.

The "sack of potatoes" version of the multi-player interaction therefore has de…nite uses. Nevertheless there will be situations in which decisions do become interlinked. It will therefore be useful to at least sketch out how a truly multi-player version of the model might look.

Let us assume that the players are indexed from 1 to n. Each player, as before, has a choice how to balance production and claiming. The production functions are given by

$$Y_{A_i} = c_i \text{®}_i W_i$$

where c_i and W_i are, as before, individual productivity and Wealth parameters. ®_i is the proportion of resources devoted to production. The appropriation functions are

$$g_i = \frac{s_i^m}{\mathbf{P}_{j=1}^{n} s_j^m}$$

where

$$s_i = (1 \text{ ¡ } \text{®}_i)\, f_{A_i} W_i$$

and f_{A_i} is the individual claiming efficiency parameter. The final payoff to player i will be given by

$$Y_i = g_i \left(\sum_{j=1} c_j \alpha_j W_j \right)$$

As before it will be useful to express many of these relationships in relative terms. Taking one of the players (say player n) as the basis for comparison, we can reparameterise the model, letting

$$k_i = \frac{W_i}{W_n}, \quad p_i = \frac{c_i}{c_n}, \quad f_i = \frac{f_{A_i}}{f_{A_n}}$$

The scale parameters c and W are then given by

$$c = c_n \text{ and } W = W_n$$

and the payoff to player i will be

$$Y_i = \frac{(1-\alpha_i)^m f_i^m k_i^m}{\sum_{j=1}^{n} (1-\alpha_j)^m f_j^m k_j^m} c \left(\sum_{j=1} p_i \alpha_i k_i \right) W \qquad (16)$$

It will be useful to refer to a player's opponents. We will in general denote the relevant magnitude with the subscript "$-i$", e.g. the collective share of a player's opponents is g_{-i}, i.e.

$$g_{-i} = \frac{\sum_{j \neq i} s_j^m}{\sum_{j=1}^{n} s_j^m}$$

The collective strength of a player's opponents we will denote by s_{-i} where[6]

$$s_{-i} = \left(\sum_{j \neq i} s_j^m \right)^{\frac{1}{m}}$$

so that

$$g_i = \frac{s_i^m}{s_i^m + s_{-i}^m} \text{ and } g_{-i} = \frac{s_{-i}^m}{s_i^m + s_{-i}^m}$$

[6] This means that s_{-i} depends on m, except in the very special case of a two-player game.

A full analysis of the multiplayer game is beyond the scope of this paper. We will content ourselves with some observations on the very specific case where we have n identical players of one type (peasants) and one player of a different type (a lord)[7]. We will refer to the lord as player B, while every one of the other players will be of type A. We will order the players so that player $n+1$ is B and we will express all our quantities relative to B, so that $k_i = k$, $p_i = p$, $f_i = f$ for $i = 1 \ldots n$. We will adopt our previous conventions and refer to B's production intensity as β.

The level of inequality (for interior intertemporal equilibria) in this model does not have a simple closed form solution. Instead, it is given by the solution to the equation

$$k^{-1} + n - 1 - n\frac{p}{f}k^{\frac{1}{m}} = 0$$

It is easy to show that this has a unique solution. Furthermore this solution will be such that

$$k > \frac{1}{n}\left(\frac{f}{p}\right)^{\frac{m}{m+1}}$$

This has the straightforward implication that inequality will be less pronounced if the peasant producers interact with each other also, than if they individually face up to the lord. We would therefore expect to see larger transfers in aggregate and richer lords in situations where the peasantry is numerous but disorganised.

If $\frac{f}{p} < 1$ (as it would be in our feudal case) then it follows that $k < \left(\frac{f}{p}\right)^{\frac{m}{m+1}}$, so that the lord is definitely richer than he would be if he had only one peasant. Furthermore in this case $\frac{\partial k}{\partial n} < 0$, so each additional peasant would increase the relative wealth of the lord.

Aggregate output is, in fact, larger in the case of the atomised peasantry. In the truly multi-player game, the incentives shift towards increased claiming by all players. Each peasant has to contend not only with the predations of the stronger lord, but also the possibility of yielding up some of her product to another peasant. In the equilibrium all peasants produce at exactly the same level so that there are no inter-peasant transfers. All transfers are between peasants and the lord. As seen in the previous section, however,

[7] In fact we will restrict our analysis also to interior intertemporal equilibria.

the absence of transfers does not imply the absence of costs. The larger the number of potential claimants, the larger the costs incurred in preempting such predations. This is reflected in a lower per capita output.

What happens to aggregate output if more peasants are introduced? We can show that

$$\frac{\partial Y_{AB}}{\partial n} > 0 \text{ if, and only if, } pk > m$$

Now increases in n reduce k, so it is evident that there is a critical peasant mass n^* beyond which increases in the number of peasants actually reduces aggregate output. This level is obviously a function of both the decisiveness parameter m and peasant productivity p. Larger values of p will also reduce the equilibrium k, but pk will increase with p.

The model of expanding empire by adding peasants at the margin (as in Landes's (1998, p.23) account referred to in the introduction) therefore runs up against inbuilt limits. These constraints can be slackened only if productivity can be enhanced. This constraint obviously does not apply to the "sack of potatoes" model. Empire building on the backs of a peasantry is therefore much more feasible if peasants do not interact with each other.

6 The impact of property rights

In all the cases considered thus far the aggregate output has been split purely according to the relative claiming efficiencies of the players. One of the implications of this is that the less productive player can always forcibly extract resources simply by being sufficiently vociferous. Being bloody-minded helps in this model. Indeed, in many circumstances it enables one to walk off with the lion's share.

Obviously this could not happen if there were perfect property rights. In that case every player would get exactly what he or she produced and there would be no claiming. Between these polar cases of absolutely inviolate property rights and shares decided purely on claiming strength, there are a number of very interesting intermediate situations[8].

The simplest way of incorporating restrictions on the predations of the other player is to provide each player with an activity which is not subject to

[8] Indeed in situations where there are strong property rights, establishing the existence of such a right will often take up energy. This can be seen as a type of claiming activity.

predation. The way one might think about this, is that it corresponds to the peasant's vegetable patch which cannot be raided by the lord. Alternatively, this might be a crop or a type of production which is subterranean, hidden from view.

Formally we assume that a fixed proportion ϕ of each agents wealth W_i can be and is invested in the secure type of production. This proportion is not subject to choice by the agent. It reflects societal norms, values and institutions such as the prevailing legal framework. Many of the cultural factors which influence m will also have an impact on ϕ. For example, a strong aversion to inequality may yield a low m, but might also make larger portions of each agent's production subject to redistributive pressure, i.e. it might erode ϕ. On the other hand primitive military technology (low m) would probably tend to increase ϕ.

We might assume that production in the secure sector occurs with exactly the same technology as production in the unprotected sector. This, however, is a restrictive assumption. It may be possible that the cost of secure production is that production is less efficient. Root crops, for example, while they may be hidden from view may yield less than cereals. Alternatively the protected form of production may be more efficient, simply because it is protected.

In general we will therefore assume that production in the protected sector will occur with a constant returns to scale technology with a productivity factor r_i, so that the payoff to production in the secure sector will be given by

$$Y_{s_i} = r_i \phi W_i$$

Since there can be no predation we have assumed that production happens with maximal efficiency.

In relation to the open sector we assume, as before, that agents have to balance claiming and productive activities. The output of production will be given by

$$Y_A = c_A \alpha (1 - \phi) W_A \text{ and } Y_B = c_B \beta (1 - \phi) W_B$$

The appropriation functions will still be given by equation 2 except that

$$\begin{aligned} s_1 &= (1 - \alpha) f_A (1 - \phi) W_A \\ s_2 &= (1 - \beta) f_B (1 - \phi) W_B \end{aligned}$$

The (1 ¡ Á) terms in the appropriation functions obligingly divide out, so that the payo¤s from the open sector will be given by

$$Y_{0_i} = (1 \text{ ¡ } Á)\, Y_i$$

where Y_1 and Y_2 are exactly as in equations 5a and 5b, i.e. the payo¤s that would have been achieved if the entire stock of resources had been invested in the open sector. Reparameterising the model as before, the …nal payo¤s to each player will be given by

$$R_A = Ár_1kW + (1 \text{ ¡ } Á)\, Y_1 \quad (17a)$$

$$R_B = Ár_2W + (1 \text{ ¡ } Á)\, Y_2 \quad (17b)$$

In this form it is clear that the payo¤s are a weighted average of the return that would be achieved on the secure asset if all resources could be devoted to producing there and the payo¤ achieved in the contest around open production if all resources were invested there.

Since Á is not a decision variable, the maximising choice of ® for p, f, m and k will be precisely as before: the reaction functions and the Cournot equilibria are all the same. One important di¤erence, however, is that the conversion of current period payo¤ into future wealth will now be given by

$$W_{A;t+1} = R_{A;t} \qquad W_{B;t+1} = R_{B;t} \quad (18)$$

and this implies that the intertemporal equilibria will be di¤erent.

The key result is that

$$k \;\mathbf{T}\; k^{¤} \text{ as } r_1 \;\mathbf{T}\; r_2$$

where $k^{¤}$ is any stable equilibrium that would have been achieved if there were no property rights, i.e. if Á = 0. Furthermore, the size of the shift (upwards or downwards) depends on the di¤erence (r_1 ¡ r_2) and on Á. The larger the term (r_1 ¡ r_2) Á, the larger the shift. An example of such a shift is depicted graphically in Figure 14

One astonishing implication of this relationship is that if $r_1 = r_2$, then granting property rights makes absolutely no di¤erence to the equilibrium outcome for any Á < 1. Even if the bulk of all production is completely protected, the di¤erences in the unprotected sector will in the long-run completely determine the distribution of wealth! The point is that theft in the "open" sector acts like a slow leak: even though the transfers every period

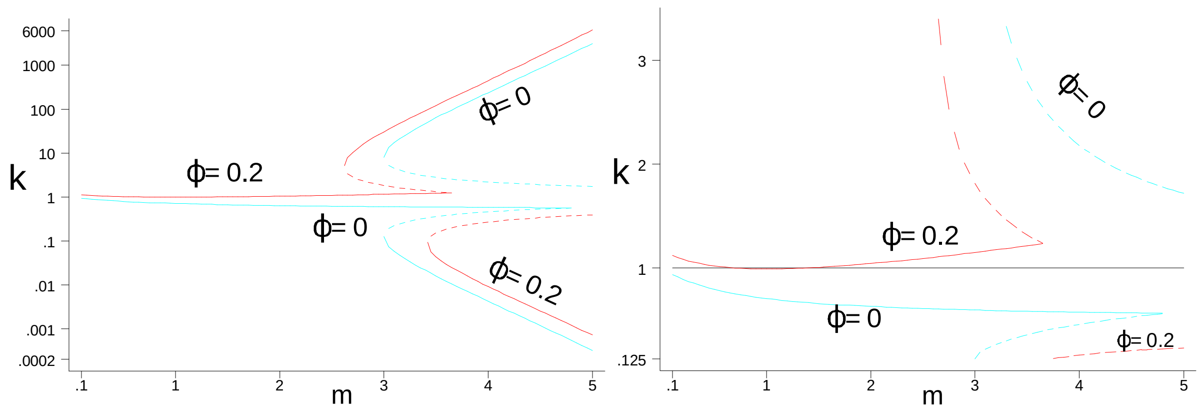

Figure 14: Property rights shift the equilibrium k upwards at all stable equilibria, provided that $r_1 > r_2$. Left panel: log scale. Right panel: detail from left panel, on a linear scale. Choice of parameters: $f = 1$, $p = 2$, $c = 2$, $r_1 = 4$, $r_2 = 2$.

may be very small, in the long-run they end up a¤ecting the overall equilibrium distribution of resources. If $r_1 = r_2$, then there are no countervailing ‡ows of income which might alter the balance determined in the contest in the open sector.

When viewed in this light, systematic theft, even if it occurs in only a small domain of the economy, could have very deleterious long-run e¤ects.

In general, one would assume, however, that a person who has a productivity advantage in the open form of production would also be more productive in the secure form of production. This would imply that our peasant or homesteader ($p > 1$, $f < 1$) may yet end up in an equilibrium with property rights, as the richer player. One way of modelling a link between the productivities r_i in the secure sector with those in the open sector is to specify them as

$$r_i = c_i^{\mu} \qquad (19)$$

This equation implies that $r_1 > r_2$ if $c_A > c_B$ (assuming $c_i > 1$). It has the additional implication that the productivity di¤erentials in the two sectors might be di¤erent. If μ is greater than one, then any productivity di¤erences in the open sector become magni…ed in the protected sector. Given that one might suppose that human capital is an inalienable asset and might

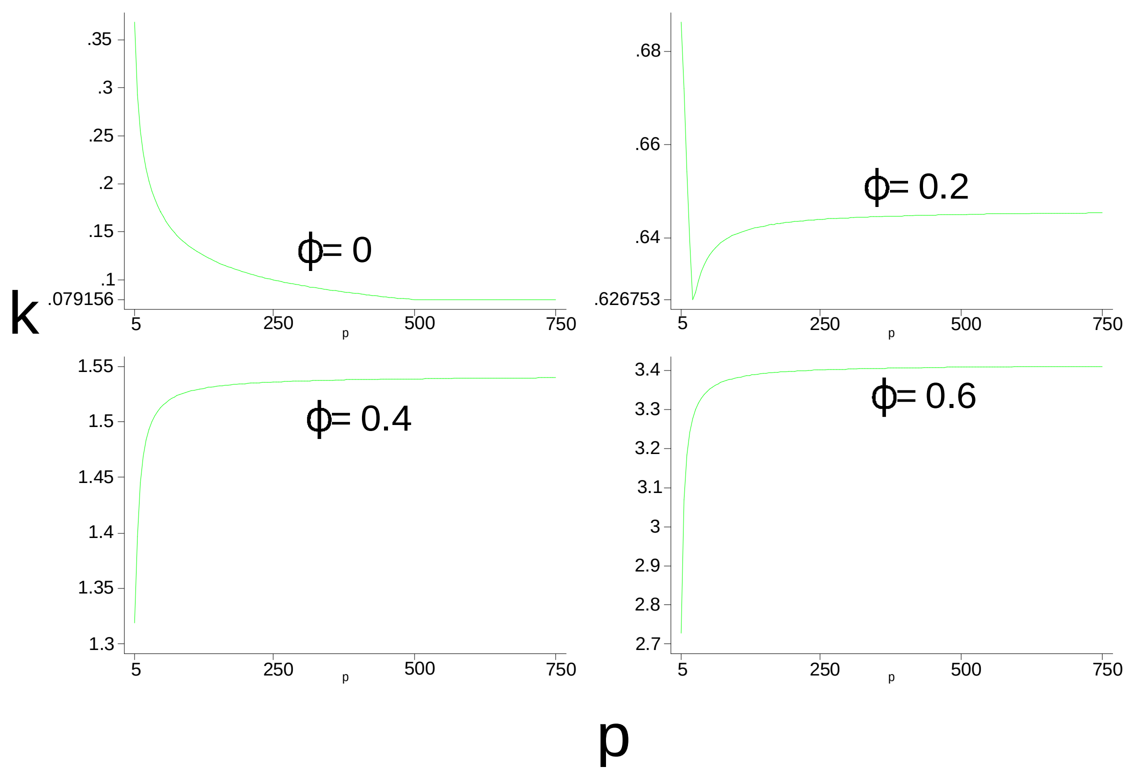

Figure 15: With sufficiently strong property rights, productivity increases might benefit the producer and not just the parasite. Choice of parameters: $f = 0{:}25$, $m = 0{:}5$, $\mu = 1$, $c = 2$.

be disproportionately exercised in the protected sector, such a widening of differentials is perhaps to be expected.

In Figures 15 and 16 we show that with stronger property rights (higher ϕ) and larger differentials in productivity in the secure sector (larger μ) we may see not only higher levels of k, but also a transformed relationship between k and p. It is possible now for productivity increases on the part of A to lead to a relative increase in A's wealth also, i.e. we may see $\frac{\partial k}{\partial p} > 0$. The top right panel in figure 15 is interesting for another reason. It shows a marked change in the relationship between k and p at the point at which player B becomes completely parasitic. To the left of this point we see that the stronger, more parasitic player manages to appropriate the benefits of the increase in productivity. Once this player has become completely parasitic,

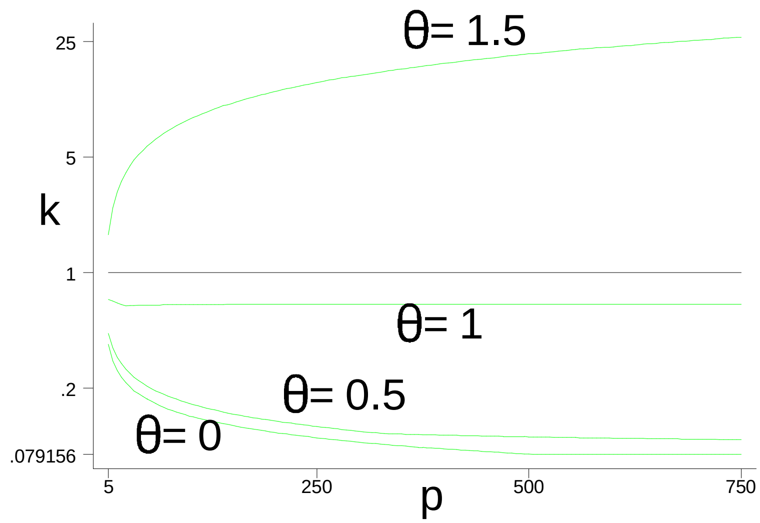

Figure 16: If Á > 0 and µ (representing. the productivity di¤erential in the secure sector) su¢ciently large, then we may have an increasing relationship between k and p, i.e. the more productive individual also becomes richer, in equilibrium. Choice of parameters: Á = 0:2, f = 0:25, m = 0:5, c = 2.

however, the maximal damage has been done and the bene…ts of productivity increases now accrue to the producer.

Indeed, although it is not altogether clear from Figure 15 one of the e¤ects of the introduction of property rights is to reduce the value of p at which player B becomes completely parasitic. When Á = 0, this transition is reached at around p = 500. When Á = 0:2, however, it occurs already at about p = 25 and when Á = 0:4 it is at about p = 8:5. The reason for this quicker onset of complete parasitism is, of course, that it is true more generally that @¯/@k < 0. As A becomes richer, the pickings to be had from parasitism increase, and the impact of A's own claiming activities increases. On both counts B …nds it necessary to increase the claiming intensity.

A positive relationship between k and p can exist in contexts other than corner solutions. Indeed, with A becoming richer, A's ability to fend o¤ B's predations increases. This is illustrated in Figure 17 which provides a closer look at the case Á = 0:4 in Figure 15. It is evident that here too there is a change in slope at the point where B becomes completely parasitic. The upward sloping relationship between k and p for the interior

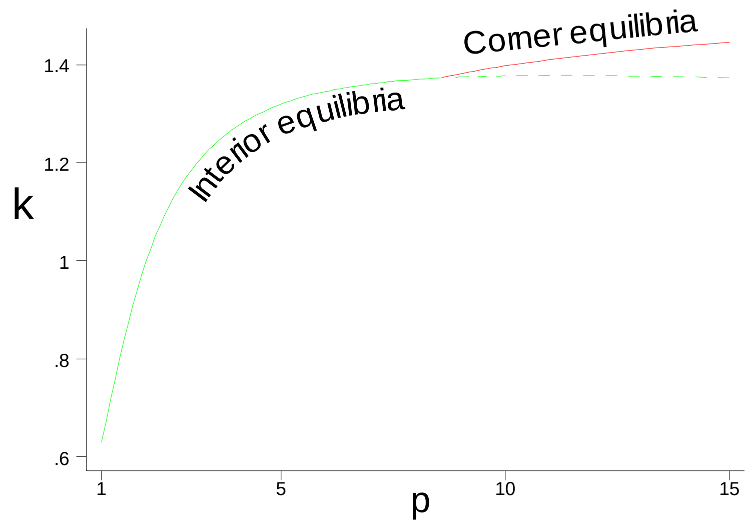

Figure 17: With Á su¢ciently high, the relationship between k and p is positive even in interior equilibria. At the point at which B becomes parasitic the slope increases. Choice of parameters: f = 0:25, m = 0:5, µ = 1, c = 2.

equilibria indicates that in this case A is su¢ciently strong to gain from her own increases in productivity. Part of the bene…t of even imperfect property rights is therefore that they provide a ‡ow of resources to the more productive player which can be used to o¤set the greater claiming e¢ciency and indeed motivation for theft on the part of the opponent.

Figure 16 indicates, however, that in the case of low property rights, the productivity di¤erentials in the protected sector have to be rather large before they have su¢ciently strong countervailing e¤ects. Indeed for the parameter choices in the model, we need the di¤erentials to be signi…cantly larger in the protected sector than in the open one in order for the more productive player to also be the richer player in equilibrium.

Because property rights a¤ect the distribution of resources in society[9] they will also a¤ect the equilibrium indices of parasitism. In Figure 18 we have graphed I_T, I_C, I_L and the growth rate, for di¤erent values of Á. For the sake of the comparison, the graphs only consider the proportion of resources in the open sector. With higher Á, the indices I_T, I_C, and I_L will shift down even further, since the con‡ict in the open sector only involves a proportion

[9] As noted above, this is true only if $r_1 \neq r_2$

of (1 ¡ Á) of all resources. Greater property rights will therefore result in even more dramatic shifts in the equilibrium level of parasitism.

Two broad trends are evident in the data: with higher Á player B becomes parasitic sooner while beyond the point where B is a complete parasite, higher levels of Á unequivocally lead to a reduction in parasitism within the open sphere. At low p, however, the quicker onset of parasitism by B suggests that societies with high property rights might actually sustain higher levels of transfers and claiming. The mediating relationship, of course, is that with inequality. Higher Á implies higher k and this in turn leads to more claiming and higher transfers. Once B has become completely parasitic, however, the bene…ts of property rights to the productive player become evident. The greater level of resources available to A, enable her to ward o¤ the e¤ects of the predation, which leads to lower transfers, i.e. lower values of I_T. This is true even if we ignore the fact that with higher Á there is a smaller portion of A's production that B can appropriate.

As far as the aggregate claiming e¤ort is concerned, we have noted above that I_C is really a weighted average of (1 ¡ ®) and (1 ¡ ¯), weighted by the relative wealth of the two players. There are two o¤-setting trends in this regard. Higher Á implies higher ® and lower ¯ in equilibrium. It also implies a relative re-weighting, with A gaining weight (due to a higher k) and B losing some. Again the direction of these shifts is clear-cut once we are in the zone where B is a complete parasite.

The lower levels of I_L record that the more productive player is also devoting a higher proportion of resources to production while the growth rate indicates that resources are shifting to the more productive individual[10].

One "feature" of Figure 18 is that it suggests that at high enough levels of p, the level of parasitism stabilises. This is certainly true for Á = 0 (as noted above in relation to Figure 4). It is also approximately true if µ = 1. If the productivity di¤erentials in the protected sector are, however, much larger, then the more productive player is able to progressively squeeze out the e¤ects of the opponent's predations. This is depicted in Figure 19, where it is also clear that the maximal level of parasitism is reached just at the point where B becomes completely parasitic. Although it may not be immediately evident from the diagrams, we can show that both I_T and I_C tend to zero in this case. Indeed, more generally I_T and I_C will approach zero as p tends to in…nity if, and only if, µ > 1.

[10] Interestingly, however, the gap between $g1_{max}$ and g_1 is wider at higher Á.

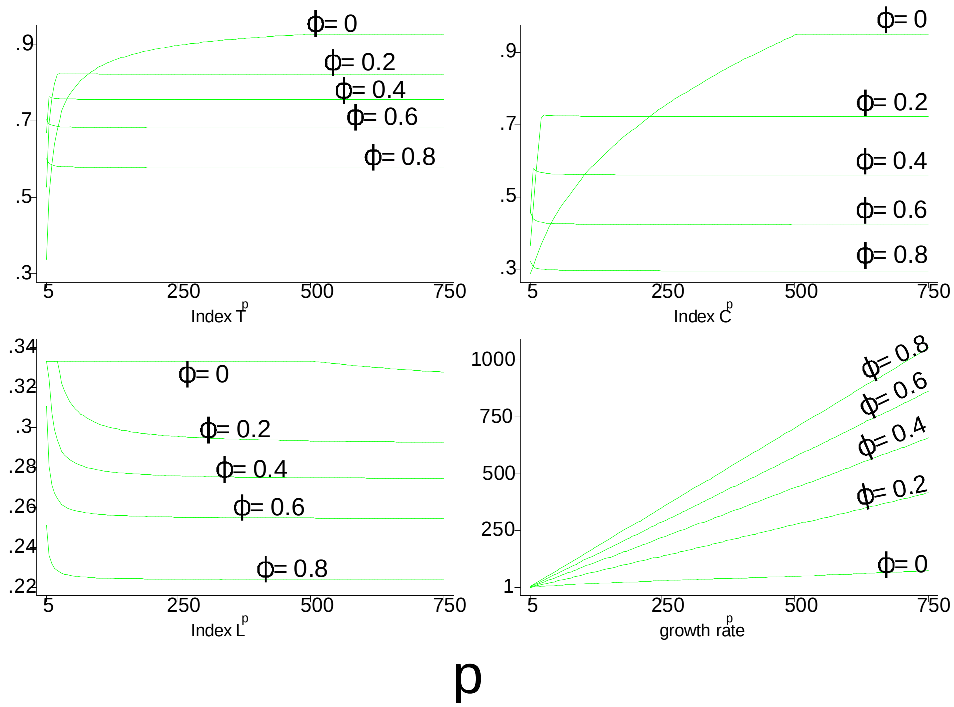

Figure 18: Indices against p with changing strength of property rights. Top left: I_T, Top right: I_C, Bottom left I_L, Bottom right: g_1. Choice of parameters: $f = 0{:}25$, $m = 0{:}5$, $\mu = 1$, $c = 2$, $W = 1$.

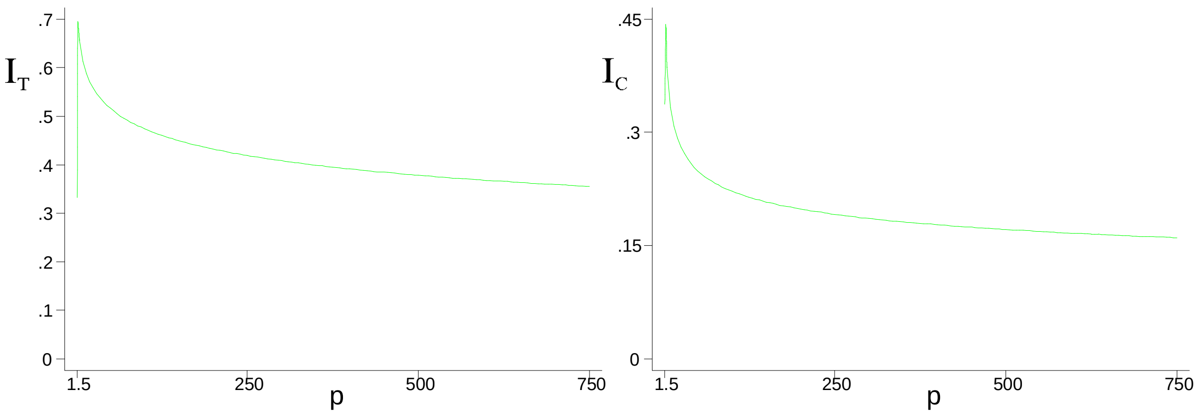

Figure 19: With higher productivity di¤erentials in the protected sector (i.e. µ > 1), the levels of transfers and of claiming (within the open sector) decrease markedly once player B has become completely parasitic and eventually tend to zero. Choice of parameters: f = 0:25, m = 0:5, Á = 0:4, µ = 1:5, c = 2, W = 1.

The reduction of the e¤ects of predation in these cases is also a process of marginalisation, akin to that discussed in the context of high values of m. It is the greater resources at the disposal of the more productive player which limit the impact of B's predation. .

7 The possibility of cooperation

Thus far the conclusion seems to be that predation is minimised either through very strong property rights (very high Á) or through the marginalisation of one of the players. Marginalisation can occur either through a very high decisiveness or through a combination of some property rights (i.e. Á > 0), a very high productivity di¤erential (p very large) and higher di¤erentials in the secure form of production than in the open sector (µ > 1).

In certain circumstances, however, it might be possible to achieve equilibria that are Pareto superior to the ones discussed thus far. This seems most obvious in the case of the "armed camp" scenario - the perfectly symmetrical game with moderately high decisiveness. As …gures 11 and 10 make clear, the costs associated with claiming in the symmetrical equilibrium are very high indeed. In some ways this game is very reminiscent of the prisoners' dilemma: if both players could only agree to be more cooperative than in the Cournot equilibrium, they would both be better o¤. Indeed if $m > 1$, this would be the only way in which a downward spiral of collapsing output (negative growth rates) could be avoided.

Since our intertemporal equilibria are all the outcomes of repeated interactions, the Folk Theorem would suggest that other equilibria ought to be possible. Provided that both players are su¢ciently patient, it ought to be possible to negotiate a process of mutual disarmament. Nevertheless in one important respect the situation in our model is di¤erent from that of the well analysed repeated prisoners' dilemma: the game is not a repetition of the same stage game, iterated inde…nitely. Instead, the results of each stage produce a wealth distribution which then sets the scene for the subsequent interaction.

One implication of this, is that if Á = 0, full cooperation (i.e. ® = 1 and ¯ = 1) can never be an equilibrium outcome. A defection by any player at this stage would e¤ectively remove the opponent and terminate the game immediately. It would be the most dramatic marginalisation considered yet. This possibility does not exist if Á > 0, since the player who had been "suckered" would still have the resources from the protected sector to fall back on.

If $m < 3$, then any agreed combination ® = ¯ (< 1) could be an equilibrium of the symmetrical game, with the threat of each player reverting to the Cournot strategy as the discipline to maintain this equilibrium. Note that any defection at this stage, while impoverishing the opponent would still leave the opponent with some resources. Since with $m < 3$ the intertemporal equilibrium is unique and stable, the path of the game after the defection would eventually converge back on the symmetrical equilibrium. The only question is whether the short-term higher payo¤s to the defector would outweigh the long-run losses. With su¢ciently patient players, the threshold of cooperation could be moved arbitrarily close to $(1; 1)$ and with Á > 0, even $(1; 1)$ would be a possible outcome, since any "suckered" player would have the resources from the protected sector available, to continue in the game and then to "punish" the defector by playing the Cournot strategy.

In the context of multiple equilibria, however, a strategy of mutual disarmament would have to restrict itself to outcomes which subsequent to a defection would not take the game into the basin of attraction of either aggressive equilibrium. This suggests that in the context of high decisiveness there is a limit to the amount of cooperation that could be expected. With strong property rights, the flow of resources to each player from the protected sector would have an equalising effect, so that the level of cooperation could in each case be set higher. Property rights might therfore promote more cooperative outcomes.

For interactions other than the symmetric one, more cooperative solutions than the Cournot equilibria ought to be possible as well. Points along the line joining any Cournot equilibrium to the point $(1;1)$ have the property that $\frac{1-\alpha}{1-\beta}$ is constant along them (two such paths are indicated in Figure 20). This, however, implies that the ratio of the shares in output $\frac{g_A}{g_B}$ is constant along such a line. In addition aggregate output is definitely larger hence both players would get higher payoffs along such a path. If $\phi = 0$ (as at the point indicated A in figure 20) then $k_{t+1} = \frac{g_{A;t}}{g_{B;t}}$ and the path in the diagram from A to $(1;1)$ preserves the equilibrium shares of wealth. As in the symmetrical case, therefore, it should be possible to negotiate a more cooperative solution. This would be a proportional reduction of claiming.

If $\phi > 0$, however, the situation becomes a little bit more complicated, since $k_{t+1} = \frac{r_1 \phi W_A + (1-\phi) Y_1}{r_2 \phi W_B + (1-\phi) Y_2}$ A balanced "disarmament" along the line indicated in the diagram would, as a first round effect, increase Y_1 and Y_2 proportionately. With $r_1 > r_2$ it would follow that this would reduce k[11]. This in turn would have second round effects. A's share would become smaller (admittedly of an altogether larger pie) and this in turn would lead to a further reduction in k. At the end of these adjustments, k would be smaller than it would be in the Cournot equilibrium.

A balanced reduction in claiming would therefore benefit the less productive or more parasitic player more. If A is initially richer (as it is in our diagram, due to strong enough ϕ and θ), we might consider this reduction in k as a voluntary form of redistribution, a progressive tax as it were, to buy social peace. If B, however, is initially richer, it would not lead to a more equal distribution of resources, instead it would make the richer parasite more richer still. Although both players would be better off, the parasite would be the bigger beneficiary of this "peace dividend". The transfers from A would resemble "protection money" paid out, not due to actual threats, but potential ones.

Note that in both cases the level of transfers would have gone up, despite

[11] $\frac{r_1 \phi W_A + (1-\phi) Y_1}{r_2 \phi W_B + (1-\phi) Y_2} > \frac{r_1 \phi W_A + (1-\phi) \delta Y_1}{r_2 \phi W_B + (1-\phi) \delta Y_2}$ (where $\delta > 1$) if, and only if $\frac{r_1}{r_2} k > \frac{Y_1}{Y_2}$. Now we have shown that with $\phi > 0$ and $r_1 > r_2$ we must have $k > \frac{Y_1}{Y_2}$. We will therefore definitely have $\frac{r_1}{r_2} k > \frac{Y_1}{Y_2}$.

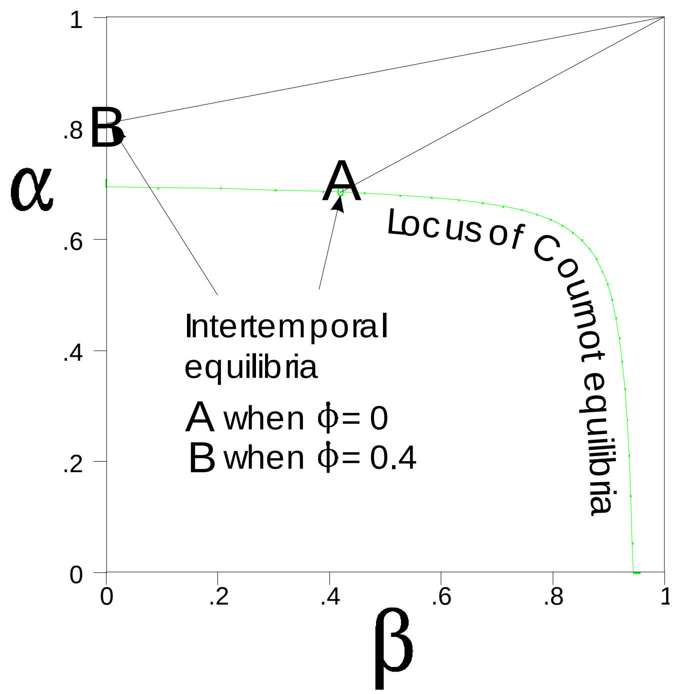

Figure 20: Points along the path from the intertemporal equilibria to (1; 1) are Pareto superior to the respective equilibrium itself. Choice of parameters: f = 0:25, m = 0:5, p = 100, c = 2, µ = 1:5.

the fact that the level of overt claiming activity would have gone down. Furthermore for the given distribution of k, the opportunity cost of claiming would go down. Nevertheless the e¤ect of the threat of predation (in these disarmament deals) is to make k less optimal than it would otherwise be for output growth. Maximum growth is, of course, achieved if resources are redistributed towards the more productive player.

8 Conclusion: The manifold e¤ects of predation

We started this paper o¤ with the problem of the di¤erent trajectories of different economies. Our analysis has con…rmed that predation can have very deleterious e¤ects on the long-run growth performance. In the absence of property rights, resources would tend to accumulate in the hands of people who have a comparative advantage in fighting and a disadvantage in production. Systems of this sort can build impressive wealth for the parasite. Indeed whole empires can be built in this way. The success of such an imperial project, however, has to be built on a system of divide and rule, because in properly interacting systems, there are inbuilt limits to how large they can be, before the effects of predation lead to economic collapse.

All of these systems grow at a slower rate than they could do, because productivity advances benefit largely the parasite and because wealth tends to be concentrated among the least productive. The worst outcomes, however, are experienced when relatively evenly matched players are pitted against each other in an environment of high decisiveness. Some of the "best" outcomes (for growth) are achieved in a context of high decisiveness through the effective marginalisation of the opponent. Nevertheless it matters greatly who is marginalised. Rancher victories are arguably less favourable to growth than homesteader ones.

The introduction of property rights paradoxically does not automatically guarantee better outcomes. Indeed, these will materialise only if the protection is stronger in areas where the more productive player also has a greater comparative advantage. This might explain the different trajectories of largely agricultural societies from more urban based ones. Perhaps the advantage of the West was not stronger property rights per se, but that these property rights were stronger in the domain of urban, industrial production. The wealth generated in this sphere could then eventually break the stranglehold of the feudal parasites. Property rights over portions of the agricultural output by contrast, might not generate sufficiently strong returns to offset the established power imbalances.

In all cases more cooperative outcomes ought to be possible. The success of such negotiated settlements depends on the players interacting with each other repeatedly. Consequently smaller societies that are not constantly overrun by migrants or marauders ought to show less overt theft or claiming activity and should therefore exhibit better growth. Nevertheless if there are positive property rights a balanced reduction of claiming would benefit the parasite more than the producer. The implication seems to be that the threat of predation might actually be more effective in achieving transfers than the predation itself. One might argue that the state (in particular the welfare state) is a formalised mechanism for policing this social pact.

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A Proofs

A.1 The basic model

Many of the properties of the model can be derived from the most general specification. In this form the payoff functions can be written as

$$
\begin{aligned}
Y_1 &= \frac{\gamma_1}{\gamma_1 + \gamma_2} Y_{AB} \\
Y_2 &= \frac{\gamma_2}{\gamma_1 + \gamma_2} Y_{AB}
\end{aligned}
$$

where $\gamma_1 = s_1^m$. We will derive many properties of the model from this most general specification and then apply them to the version with our choice of functional forms. This version is given by:

$$
Y_1 = \frac{(1-\alpha)^m f_A^m W_A^m}{(1-\alpha)^m f_A^m W_A^m + (1-\beta)^m f_B^m W_B^m} \left[c_A \alpha W_A + c_B \beta W_B\right] \quad \text{(20a)}
$$

$$
Y_2 = \frac{(1-\beta)^m f_B^m W_B^m}{(1-\alpha)^m f_A^m W_A^m + (1-\beta)^m f_B^m W_B^m} \left[c_A \alpha W_A + c_B \beta W_B\right] \quad \text{(20b)}
$$

In this form the model is obviously symmetrical for A and B. Any statement which is true in this model will remain true if we reassign variables as follows $Y_1 \to Y_2$; $Y_2 \to Y_1$, $\alpha \to \beta$, $\beta \to \alpha$, $c_A \to c_B$, $c_B \to c_A$, $f_A \to f_B$, $f_B \to f_A$, $W_A \to W_B$, $W_B \to W_A$. When we reparameterise the model such that

$$
k = \frac{W_A}{W_B}, \quad p = \frac{c_A}{c_B} \quad \text{and} \quad f = \frac{f_A}{f_B}
$$

this obvious symmetry is broken. As noted in the text the fact that W_A and c_A no longer feature in the model necessitates a reinterpretation of these parameters, so a change in W_B with constant parameter k has the implication of increasing both W_A and W_B. This implies that the parameter W_B now represents the baseline wealth of the society as a whole and c_B the baseline productivity. We drop the subscripts to reflect this reinterpretation.

The payoff equations become (as in equations 5a and 5b)

$$
\begin{aligned}
Y_1 &= \frac{(1-\alpha)^m f^m k^m}{(1-\alpha)^m f^m k^m + (1-\beta)^m} c\,(p\alpha k + \beta)\,W \\
Y_2 &= \frac{(1-\beta)^m}{(1-\alpha)^m f^m k^m + (1-\beta)^m} c\,(p\alpha k + \beta)\,W
\end{aligned}
$$

Despite the apparent asymmetry in this form of the model, the strategic choices facing A and B are obviously still symmetric. If we interchange A's and B's wealth, productivity and appropriation efficiency the optimal responses will also be interchanged.

Remark 1 Any statement about B when suitably reinterpreted will be true of A. If we make the following substitutions in any statement, the statement will continue to be true: $Y_1 \to Y_2$; $Y_2 \to Y_1$, $\alpha \to \beta$, $\beta \to \alpha$, $p \to p^{-1}$, $k \to k^{-1}$, $f \to f^{-1}$, $c \to pc$, $W \to kW$.

This can be readily checked in the equations above. We will rely on this property to derive the results once and then reinterpret them for the symmetrical case.

A.2 Properties of the payoff and reaction functions

Lemma 2 We have $(1-\alpha)\frac{\partial\gamma_1}{\partial\alpha} = -m\gamma_1$ and $(1-\beta)\frac{\partial\gamma_2}{\partial\beta} = -m\gamma_2$

Proof. This follows from the fact that $\frac{\gamma_1}{\gamma_1+\gamma_2}$ is homogeneous of degree zero in $(1-\alpha)$ and $(1-\beta)$ and that γ_1 and γ_2 are not functions of β and α respectively. By applying Euler's theorem to $\frac{\gamma_1}{\gamma_1+\gamma_2}$ we get the identity

$$(1-\alpha)\gamma_2\frac{\partial\gamma_1}{\partial\alpha} \equiv (1-\beta)\gamma_1\frac{\partial\gamma_2}{\partial\beta}$$

This can hold only if

$$(1-\alpha)\frac{\partial\gamma_1}{\partial\alpha} \equiv -\theta\gamma_1 \text{ and } -\theta\gamma_2 = (1-\beta)\frac{\partial\gamma_2}{\partial\beta}$$

for some constant $\theta \geq 0$. In our case this constant is m. ■

Proposition 3 $\frac{\partial Y_1}{\partial\beta} > 0$ and $\frac{\partial Y_2}{\partial\alpha} > 0$

Proof. We prove the case for $\frac{\partial Y_1}{\partial\beta}$. The other case is analogous. Now

$$\frac{\partial Y_1}{\partial\beta} = \frac{-\gamma_1\frac{\partial}{\partial\beta}(\gamma_2)}{[\gamma_1+\gamma_2]^2}Y_{AB} + \frac{\gamma_1}{\gamma_1+\gamma_2}\frac{\partial Y_{AB}}{\partial\beta}$$

By our assumptions $\frac{\partial}{\partial\beta}(\gamma_2) < 0$ and $\frac{\partial Y_{AB}}{\partial\beta} > 0$ so the result follows immediately. ■

Proposition 4 For a given value of β we have that $\frac{\partial Y_1}{\partial \alpha} \gtrless 0$ as $Y_2 \lessgtr \frac{(1-\alpha)}{m}\frac{\partial Y_{AB}}{\partial \alpha}$. Similarly for a given value of α we have that $\frac{\partial Y_2}{\partial \beta} \gtrless 0$ as $Y_1 \lessgtr \frac{(1-\beta)}{m}\frac{\partial Y_{AB}}{\partial \beta}$

Proof. We know that

$$\frac{\partial Y_1}{\partial \alpha} = \frac{\gamma_2 \frac{\partial}{\partial \alpha}(\gamma_1)}{[\gamma_1 + \gamma_2]^2} Y_{AB} + \frac{\gamma_1}{\gamma_1 + \gamma_2}\frac{\partial Y_{AB}}{\partial \beta}$$

Using the lemma 2, excluding the case $\alpha = 1$ and simplifying we get

$$\frac{\partial Y_1}{\partial \alpha} = \frac{\gamma_1}{\gamma_1 + \gamma_2}\left[\frac{-m}{(1-\alpha)} Y_2 + \frac{\partial Y_{AB}}{\partial \alpha}\right] \tag{21}$$

If we exclude the possibility that $\frac{\gamma_1}{\gamma_1+\gamma_2} = 0$ we have the result. ■

Theorem 5 Given our choice of functional form, and for fixed values of the parameters $\beta; k; p; f; m; c; W$ the payoff function $Y_1(\alpha)$ can have one of the following characteristics:

- Y_1 increases from $\alpha = 0$ until it reaches its maximum from where it decreases to the global minimum at $\alpha = 1$.
- Y_1 decreases monotonically from $\alpha = 0$ to the global minimum at $\alpha = 1$.

  Similarly for fixed values of $\alpha; k; p; f; m; c; W$ the payoff function $Y_2(\beta)$ can either reach a maximum in the interior of $(0; 1)$, or decrease monotonically on $[0; 1)$.

Proof. We will prove the case for $Y_2(\beta)$. The proof for $Y_1(\alpha)$ follows by symmetry. By proposition 4 we have that $\frac{\partial Y_2}{\partial \beta} \gtrless 0$ as $Y_1 \lessgtr \frac{(1-\beta)}{m}\frac{\partial Y_{AB}}{\partial \beta}$. Substituting in we get $\frac{\partial Y_2}{\partial \beta} \gtrless 0$ as

$$\frac{(1-\alpha)^m f^m k^m}{(1-\alpha)^m f^m k^m + (1-\beta)^m} c(p\alpha k + \beta) W \lessgtr \frac{(1-\beta)}{m} cW$$

Now note that if $\beta = 1$ the right hand side is zero, while the left hand side is equal to $c(p\alpha k + \beta)W$ which is positive[12]. Consequently $\frac{\partial Y_2}{\partial \beta}$ is negative near $\beta = 1$. Furthermore the left-hand side is monotonically increasing in

[12] Provided that $a \neq 1$, in which case, the left-hand side is not defined

β. We therefore need to consider only what happens at $\beta = 0$. If we have $\frac{(1-\alpha)^m f^m k^m}{(1-\alpha)^m f^m k^m+1} c(p\alpha k) W < \frac{1}{m} cW$, then there will be a unique $\beta_0 \in (0;1)$ for which the left-hand side is equal to the right-hand side, i.e. $\frac{\partial Y_2}{\partial \beta} = 0$. This point is clearly a maximum since $\frac{\partial Y_2}{\partial \beta}$ is increasing on $(0;\beta_0)$ and decreasing on $(\beta_0;1)$ ■

Corollary 6 It is clear that B's reaction function will be given by the locus of interior maxima of $Y_2(\beta)$ i.e. by the solutions to the equation $\frac{\partial Y_2}{\partial \beta} = 0$ (where these exist) and otherwise by $\beta = 0$.

Proposition 7 $Y_1(\alpha)$ will have an interior maximum in a neighbourhood of $\beta = 0$ and of $\beta = 1$. It will have an interior maximum for every $\beta \in [0;1)$ if $pk \geq m$.

$Y_2(\beta)$ will have an interior maximum in a neighbourhood of $\alpha = 0$ and of $\alpha = 1$. It will have an interior maximum for every $\alpha \in [0;1)$ if $pkm \leq 1$.

Proof. (of the case for $Y_2(\beta)$) We have noted in the proof of Theorem 5 above that the equation $\frac{\partial Y_2}{\partial \beta} = 0$ has a solution only if

$$\frac{(1-\alpha)^m f^m k^m}{(1-\alpha)^m f^m k^m + 1}(p\alpha k) < \frac{1}{m} \tag{22}$$

This inequality will certainly hold in a neighbourhood of $\alpha = 0$ and of $\alpha = 1$. It will hold for every $\alpha \in [0;1]$ if $pkm \leq 1$. ■

Remark 8 $Y_2(\beta)$ will have an interior maximum if $Y_1(\alpha;0) \leq \frac{cW}{m}$.

Proof. Inequality 22 above can, of course be written equivalently as $\frac{(1-\alpha)^m f^m k^m}{(1-\alpha)^m f^m k^m+1} c(p\alpha k) W < \frac{cW}{m}$. The left hand side is the payoff to Player A if $\beta = 0$, i.e. it is $Y_1(\alpha|\beta = 0)$ ■

Proposition 9 If there is an interior solution to the equation $\frac{\partial Y_1}{\partial \alpha} = 0$ it will be a maximum if $\frac{\partial^2 Y_{AB}}{\partial \alpha^2} < 0$. If $\frac{\partial^2 Y_{AB}}{\partial \alpha^2} > 0$ then the solution may be a maximum or minimum.

Proof. Assume that for $\beta = \beta_0$ we have $\frac{\partial Y_1}{\partial \alpha} = 0$ at $\alpha = \alpha_0$ i.e. $Y_2(\alpha_0;\beta_0) = \frac{(1-\alpha_0)}{m}\frac{\partial Y_{AB}}{\partial \alpha}$. At this solution we have

$$\frac{\partial}{\partial \alpha}\left(\frac{\partial Y_1}{\partial \alpha}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\gamma_1}{\gamma_1+\gamma_2}\right) \cdot \left[\frac{-m}{(1-\alpha)}Y_2 + \frac{\partial Y_{AB}}{\partial \alpha}\right] + \frac{\gamma_1}{\gamma_1+\gamma_2}\left[\frac{-m}{(1-\alpha)^2}Y_2 - \frac{m}{(1-\alpha)}\frac{\partial Y_2}{\partial \alpha} + \frac{\partial^2 Y_{AB}}{\partial \alpha^2}\right]$$

The first term on the right hand side is zero, since we are evaluating the expression at $(\alpha_0;\beta_0)$, i.e.

$$\frac{\partial^2 Y_1}{\partial\alpha^2} = \frac{\gamma_1}{\gamma_1+\gamma_2}\left[\frac{-m}{(1-\alpha)^2}Y_2 - \frac{m}{(1-\alpha)}\frac{\partial Y_2}{\partial\alpha} + \frac{\partial^2 Y_{AB}}{\partial\alpha^2}\right]$$

We have shown that $\frac{\partial Y_2}{\partial\alpha} > 0$, so if $\frac{\partial^2 Y_{AB}}{\partial\alpha^2} < 0$ we will undoubtedly have $\frac{\partial^2 Y_1}{\partial\alpha^2} < 0$. If $\frac{\partial^2 Y_{AB}}{\partial\alpha^2} > 0$ then the sign of the expression in square brackets is indeterminate.

Note that if Y_{AB} is separable, i.e. if $Y_{AB} = Y_A + Y_B$ then $\frac{\partial^2 Y_{AB}}{\partial\alpha^2} = \frac{\partial^2 Y_A}{\partial\alpha^2}$. If Y_A is a concave function then $\frac{\partial^2 Y_A}{\partial\alpha^2} < 0$ and we come to the conclusion that any stationary point on the payoff function is a global maximum. ■

Remark 10 By proposition 4 the conditions $\frac{\partial Y_1}{\partial\alpha} = 0$ and $\frac{\partial Y_2}{\partial\beta} = 0$ can be written respectively as

$$\frac{(1-\beta)^m}{(1-\alpha)^m f^m k^m + (1-\beta)^m}(p\alpha k + \beta) = \frac{(1-\alpha)pk}{m} \tag{23}$$

$$\frac{(1-\alpha)^m f^m k^m}{(1-\alpha)^m f^m k^m + (1-\beta)^m}(p\alpha k + \beta) = \frac{(1-\beta)}{m} \tag{24}$$

Definition 11 Let $\beta = r_B(\alpha)$ be the locus of global maxima of the payoff function $Y_2(\beta)$ for fixed values of $k; p; f; m; c; W$ defined for $\alpha \in [0;1)$. Let $\alpha = r_A(\beta)$ be the locus of global maxima of the payoff function $Y_1(\alpha)$ for fixed values of $k; p; f; m; c; W$ defined on $[0;1)$.

Proposition 12 The slope of the reaction function $\alpha = r_A(\beta)$ at any interior point will be given by

$$\frac{\partial\alpha}{\partial\beta} = -\frac{\frac{-m}{(1-\alpha)}\frac{\partial Y_2}{\partial\beta} + \frac{\partial^2 Y_{AB}}{\partial\beta\partial\alpha}}{\frac{-m}{(1-\alpha)^2}Y_2 - \frac{m}{(1-\alpha)}\frac{\partial Y_2}{\partial\alpha} + \frac{\partial^2 Y_{AB}}{\partial\alpha^2}}$$

provided that $\frac{-m}{(1-\alpha)^2}Y_2 - \frac{m}{(1-\alpha)}\frac{\partial Y_2}{\partial\alpha} + \frac{\partial^2 Y_{AB}}{\partial\alpha^2} \neq 0$. The slope of the reaction function $\beta = r_B(\alpha)$ at any interior point will be given by

$$\frac{\partial\beta}{\partial\alpha} = -\frac{\frac{-m}{(1-\beta)}\frac{\partial Y_1}{\partial\alpha} + \frac{\partial^2 Y_{AB}}{\partial\alpha\partial\beta}}{\frac{-m}{(1-\beta)^2}Y_1 - \frac{m}{(1-\beta)}\frac{\partial Y_1}{\partial\beta} + \frac{\partial^2 Y_{AB}}{\partial\beta^2}}$$

provided that $\frac{-m}{(1-\beta)^2}Y_1 - \frac{m}{(1-\beta)}\frac{\partial Y_1}{\partial\beta} + \frac{\partial^2 Y_{AB}}{\partial\beta^2} \neq 0$.

Proof. We make use of the implicit function theorem. Since we are at an interior point, we may assume that $\alpha = r_A(\beta)$ can be defined implicitly by the equation $\frac{\partial Y_1}{\partial \alpha} = 0$ and in particular by:

$$\frac{-m}{(1-\alpha)}Y_2 + \frac{\partial Y_{AB}}{\partial \alpha} \approx 0$$

Let $F(\alpha;\beta) = \frac{-m}{(1-\alpha)}Y_2 + \frac{\partial Y_{AB}}{\partial \alpha}$. Then

$$\frac{\partial \alpha}{\partial \beta} = -\frac{F_\beta}{F_\alpha}$$

This will be valid provided that $F_\alpha \neq 0$. Now $F_\alpha = \frac{-m}{(1-\alpha)^2}Y_2 - \frac{m}{(1-\alpha)}\frac{\partial Y_2}{\partial \alpha} + \frac{\partial^2 Y_{AB}}{\partial \alpha^2}$. Furthermore $F_\beta = \frac{-m}{(1-\alpha)}\frac{\partial Y_2}{\partial \beta} + \frac{\partial^2 Y_{AB}}{\partial \beta \partial \alpha}$. The other result follows by symmetry. ■

Theorem 13 B's reaction function $r_B(\alpha)$ will have one of three possible shapes:

1. r_B increases monotonically over the interval $[0;1)$ and $r_B(\alpha) > \frac{1}{m+1}$ for all α.
2. r_B monotonically decreases over the interval $[0;\bar{\alpha}]$ and then monotonically increases on $[\bar{\alpha};1)$
3. r_B monotonically decreases over $[0;\underline{\alpha}_1]$ to zero, $r_B = 0$ for $\alpha \in [\underline{\alpha}_1;\underline{\alpha}_2]$ and r_B increases monotonically on $[\underline{\alpha}_2;1)$.

   In all case $r_B(0) > \frac{1}{m+1}$ and $r_B(\alpha) \to 1$ as $\alpha \to 1$

An analogous result applies for A's reaction function $r_A(\beta)$.

Proof. We note (by proposition 7) that at $\alpha = 0$ we have an interior maximum. The condition $\frac{\partial Y_2}{\partial \beta} = 0$ can be written in the form of equation 24 Substituting $\alpha = 0$ into this expression and simplifying, we get that

$$(1-\beta)^{m+1} = f^m k^m (\beta(m+1) - 1)$$

Equality can hold only if the right hand side is positive, i.e. $\beta > \frac{1}{m+1}$. This establishes that $r_B(0) > \frac{1}{m+1}$.

We have also observed that near $\alpha = 1$ we will also have an interior maximum. Note that $\lim_{\alpha \to 1} \frac{(1-\alpha)^m f^m k^m}{(1-\alpha)^m f^m k^m + (1-\beta)^m} (p\alpha k + \beta) = 0$ so in order for equality to hold in equation 24, we need $\lim_{\alpha \to 1} \beta = 1$.

The rest of the theorem follows by considering the slopes of the reaction functions in the interior of the unit square. By proposition 12 we have

$$\frac{\partial \beta}{\partial \alpha} = - \frac{\frac{-m}{(1-\beta)} \frac{\partial Y_1}{\partial \alpha} + \frac{\partial^2 Y_{AB}}{\partial \alpha \partial \beta}}{\frac{-m}{(1-\beta)^2} Y_1 - \frac{m}{(1-\beta)} \frac{\partial Y_1}{\partial \beta} + \frac{\partial^2 Y_{AB}}{\partial \beta^2}}$$

With our choice of functional form $\frac{\partial^2 Y_{AB}}{\partial \beta^2} = 0$, so the denominator is definitely negative and non-zero. Furthermore $\frac{\partial^2 Y_{AB}}{\partial \alpha \partial \beta} = 0$. It follows that the sign of $\frac{\partial \beta}{\partial \alpha}$ is opposite to that of $\frac{\partial Y_1}{\partial \alpha}$. By proposition 4 $\frac{\partial Y_1}{\partial \alpha} \gtreqless 0$ as $Y_2 \lesseqgtr \frac{(1-\alpha)}{m} \frac{\partial Y_{AB}}{\partial \alpha}$. Assume now that $\frac{\partial \beta}{\partial \alpha} > 0$ at some point $(\alpha_0; r_B(\alpha_0))$ on the reaction function, i.e. $\frac{\partial Y_1}{\partial \alpha} < 0$ at α_0. We must have

$$Y_2(\alpha_0; r_B(\alpha_0)) > \frac{(1-\alpha_0)}{m} c(pk) W \tag{25}$$

The total derivative of Y_2 with respect to α at this point is $\frac{d}{d\alpha} Y_2 = \frac{\partial Y_2}{\partial \alpha} + \frac{\partial Y_2}{\partial \beta} \frac{\partial \beta}{\partial \alpha}$. Since $\frac{\partial Y_2}{\partial \beta} = 0$ at $(\alpha_0; r_B(\alpha_0))$ the left-hand side of inequality 25 will unambiguously increase with an increase in α. The right hand side will unambiguously decrease. Consequently the sign of $\frac{\partial Y_1}{\partial \alpha}$ will remain negative. It follows that if $\frac{\partial \beta}{\partial \alpha} > 0$ at any α_0 then for all $\alpha > \alpha_0$ we must also have $\frac{\partial \beta}{\partial \alpha} > 0$.

Now observe that for α near 1 we have shown that there must be interior points on the reaction function. Furthermore in this region inequality 25 must hold. Consequently the reaction function r_B must be increasing on some interval to the left of $\alpha = 1$.

It remains to consider what happens at $\alpha = 0$. It is possible that we have $Y_2(0; r_B(0)) > \frac{1}{m} c(pk) W$. In that case the reaction function will increase monotonically from $\alpha = 0$ for all $\alpha \in [0; 1)$. If $Y_2(0; r_B(0)) = \frac{1}{m} c(pk) W$ then $\frac{\partial \beta}{\partial \alpha} = 0$ at $\alpha = 0$. However we have $\frac{d}{d\alpha} Y_2(\alpha; r_B(\alpha)) > 0$, so $\frac{\partial Y_1}{\partial \alpha}$ evaluated at $(\alpha; r_B(\alpha))$ will be negative to the right of $\alpha = 0$. Again the reaction function will increase monotonically.

If $Y_2(0; r_B(0)) < \frac{1}{m} c(pk) W$ then the reaction function will decrease from $\alpha = 0$. There are now two possibilities: either the reaction function will reach a stationary point at $\alpha = \bar{\alpha}$ in the interior of the unit square, or it continues decreasing until it reaches the boundary at $\underline{\alpha}_1$, i.e. $r_B(\underline{\alpha}_1) = 0$.

If we have $\frac{\partial\beta}{\partial\alpha} = 0$ at $(\alpha; r_B(\alpha))$ with $r_B(\alpha) > 0$, then we must have $Y_2(\alpha; r_B(\alpha)) = \frac{1-\alpha}{m} c(pk) W$. By the previous arguments it follows that to the right of α we must have $\frac{\partial\beta}{\partial\alpha} > 0$ and hence the reaction function must increase towards one on the interval $(\alpha; 1)$.

Assume now that at $\underline{\alpha}_1$ we have $r_B(\underline{\alpha}_1) = 0$, with $Y_2(\underline{\alpha}_1; 0) \leq \frac{1-\underline{\alpha}_1}{m} c(pk) W$. If we have equality the previous case applies, so assume that $Y_2(\underline{\alpha}_1; 0) < \frac{1-\underline{\alpha}_1}{m} c(pk) W$. This implies by proposition 4 that $\frac{\partial Y_1}{\partial\alpha} > 0$. The payoff function Y_1 for $\beta = 0$ is therefore still increasing at $\alpha = \underline{\alpha}_1$. It will reach its maximum at $\alpha = \alpha^*$.

Note that because $r_B(\underline{\alpha}_1) = 0$, the solution to the equation $\frac{\partial Y_2}{\partial\beta} = 0$ is given by $\beta = 0$. It follows (again from proposition 4) that $Y_1(\underline{\alpha}_1; 0) = \frac{cW}{m}$. Since α^* maximises the payoff function $Y_1(\alpha; 0)$, we must have $Y_1(\alpha^*; 0) > \frac{cW}{m}$. Since Y_1 decreases monotonically on $(\alpha^*; 1)$ towards zero, there must be a unique $\alpha = \underline{\alpha}_2$ on this interval such that $Y_1(\underline{\alpha}_2; 0) = \frac{cW}{m}$. Since Y_1 is first increasing and then decreasing on $(\underline{\alpha}_1; \underline{\alpha}_2)$ it follows that $Y_1(\alpha; 0) > \frac{cW}{m}$ for all $\alpha \in (\underline{\alpha}_1; \underline{\alpha}_2)$. This, however, proves that $\frac{\partial Y_2}{\partial\beta} < 0$ at $\beta = 0$ for all $\alpha \in (\underline{\alpha}_1; \underline{\alpha}_2)$. It follows by Theorem 5 that there will be no interior maxima of Y_2 on this interval, i.e. $r_B(\alpha) = 0$ for $\alpha \in (\underline{\alpha}_1; \underline{\alpha}_2)$.

Now consider what happens to the reaction function at $\alpha = \underline{\alpha}_2$. We know that $\frac{\partial Y_1}{\partial\alpha} < 0$ at $\underline{\alpha}_2$ so $Y_1(\alpha; 0) < \frac{cW}{m}$ to the right of $\underline{\alpha}_2$. We will therefore again have interior solutions. Furthermore since $\frac{\partial Y_1}{\partial\alpha} < 0$ at $\underline{\alpha}_2$ we must have $\frac{\partial\beta}{\partial\alpha} > 0$ to the right of $\underline{\alpha}_2$. The reaction function is therefore monotonically increasing on this interval. ■

Corollary 14 If r_B is of the third type, then r_A will be of the first type and if r_B is of the first type, then r_A will be of the third type, and conversely.

Proof. We have noted in the proof above that if r_B is of the third type, then A's optimal response to $\beta = 0$ i.e. α^* is such that $\alpha^* \in (\underline{\alpha}_1; \underline{\alpha}_2)$. Furthermore at $(\alpha^*; 0)$ we have $\frac{\partial Y_2}{\partial\beta} < 0$. By proposition 12 it follows that A's reaction function r_A has a positive slope at $(\alpha^*; 0)$, i.e. $\frac{\partial\alpha}{\partial\beta} > 0$. This, however, implies that it is of type 1.

If r_B is of the first type, then we have at the point $(0; r_B(0))$ that $\frac{\partial Y_1}{\partial\alpha} \leq 0$. This, however, implies (by Theorem 5) that for $\beta^* = r_B(0)$ the payoff function Y_1 has its maximum on the boundary, i.e. at $\alpha = 0$. This implies that r_A is of the third type.

The converse results hold by symmetry. ■

A.3 The Cournot equilibria

Proposition 15 If $\frac{\partial^2 Y_{AB}}{\partial\beta\partial\alpha} = 0$, then if the two reaction functions $r_B(\alpha)$ and $r_A(\beta)$ intersect in the interior of $[0;1) \times [0;1)$ they will do so at right angles.

Proof. This follows from 12. If we set $\frac{\partial^2 Y_{AB}}{\partial\beta\partial\alpha} = 0$, then along A's reaction function we have $\frac{\partial\alpha}{\partial\beta} = 0$ if and only if $\frac{\partial Y_2}{\partial\beta} = 0$, while along B's reaction function we have $\frac{\partial\beta}{\partial\alpha} = 0$, if and only if, $\frac{\partial Y_1}{\partial\alpha} = 0$. Now at the point of intersection of r_A and r_B we must have both $\frac{\partial Y_2}{\partial\beta} = 0$ and $\frac{\partial Y_1}{\partial\alpha} = 0$. It follows that both $\frac{\partial r_B(\alpha)}{\partial\alpha} = 0$ and $\frac{\partial r_A(\beta)}{\partial\beta} = 0$. So while the one reaction curve is horizontal, the other will be vertical. ■

Definition 16 Let $\alpha^* = r_A(0)$, i.e. α^* is the unique solution in $(0;1)$ of $(1-\alpha)^{m+1} f^m k^m = \alpha(m+1) - 1$

Let $\beta^* = r_B(0)$, i.e. β^* is the unique solution in $(0;1)$ of $(1-\beta)^{m+1} = f^m k^m (\beta(m+1) - 1)$.

Theorem 17 Let $\beta = r_B(\alpha; k; p; f; m; c; W)$ and $\alpha = r_A(\beta; k; k; p; f; m; c; W)$ be B's and A's reaction functions respectively. Then

1. $\frac{\partial r_B(\alpha)}{\partial\alpha} > 0$ for all $\alpha \in [0;1)$ if, and only if, $r_A(\beta^*) = 0$. The Cournot solution is therefore $(0; \beta^*)$

2. r_B has an interior turning point at $\hat{\alpha}$ if, and only if, r_A has a turning point at $\delta = r_B(\hat{\alpha})$. The point $\left(\hat{\alpha}; \delta\right)$ is therefore the intersection of r_B and r_A and will be the Cournot solution. We have $\frac{\partial r_B(\alpha)}{\partial\alpha} = 0$ at $\left(\hat{\alpha}; \delta\right)$ and $\frac{\partial r_A(\beta)}{\partial\beta} = 0$, i.e. the reaction curves intersect at right angles.

3. $r_B(\alpha^*) = 0$ if, and only if, $\frac{\partial r_A(\beta)}{\partial\beta} > 0$ for all $\beta \in [0;1)$. The Cournot solution is therefore $(\alpha^*; 0)$.

Proof. The results all follow from corollary 14 and proposition 15. The corollary establishes cases 1 and 3. As far as case 2 is concerned, we note that at an interior Cournot equilibrium the two reaction curves meet at right angles. The implication of this is that at any point $\left(\hat{\alpha}; \delta\right)$ which satisfies both the condition that $\frac{\partial Y_1}{\partial\alpha} = 0$ and $\frac{\partial Y_2}{\partial\beta} = 0$ we must have along r_B that $\frac{\partial\beta}{\partial\alpha} = 0$. Similarly we will have along r_A that $\frac{\partial\alpha}{\partial\beta} = 0$.

Conversely, however, if there should be any point $(\alpha_0;\beta_0)$ along the reaction curve r_B where $\frac{\partial\beta}{\partial\alpha}=0$ it follows that this point will satisfy the equation $\frac{\partial Y_1}{\partial\alpha}=0$. Consequently it will fall onto the curve r_A, i.e. it is the point of intersection of the reaction functions. This proves case 2. ■

Corollary 18 Let $\left(\alpha;\delta\right)$ be any Cournot equilibrium. We must have $r_A(\beta) \geq \alpha$ for all $\beta \in [0;\infty)$ and $r_B(\alpha) \geq \delta$ for all $\alpha \in [0;\infty)$.

Proof. In the case of interior equilibria, the reaction curves intersect at their respective turning points. These are the global minima of these curves. In the case of a corner solution $(0;\beta^*)$ or $(\alpha^*;0)$ it is trivially true for the "corner", that respectively $r_A(\beta) \geq 0$ or $r_B(\alpha) \geq 0$. The other reaction curve will in this case be monotonically increasing and hence $r_B(\alpha) \geq \beta^*$ and $r_B(\alpha) \geq \alpha^*$ respectively. ■

This result states that in some sense the Cournot equilibrium is maximally uncooperative - maximally uncooperative among all rational responses.

Proposition 19 Assume that B's reaction function $\beta = r_B(\alpha)$ is defined on some neighbourhood in the interior of $[0;\infty) \times [0;\infty)$. Assume further that A's reaction function $\alpha = r_A(\beta)$ is defined on the same neighbourhood. If these reaction functions intersect in the interior, then at this interior Cournot equilibrium we will have

$$\frac{\gamma_1}{\gamma_2} = \frac{(1-\beta)\frac{\partial Y_{AB}}{\partial\beta}}{(1-\alpha)\frac{\partial Y_{AB}}{\partial\alpha}} \tag{26}$$

$$Y_{AB} = \frac{(1-\alpha)}{m}\frac{\partial Y_{AB}}{\partial\alpha} + \frac{(1-\beta)}{m}\frac{\partial Y_{AB}}{\partial\beta} \tag{27}$$

Proof. By proposition 4 the reaction functions must satisfy (respectively)

$$\frac{\gamma_2}{\gamma_1+\gamma_2}Y_{AB} = \frac{(1-\alpha)}{m}\frac{\partial Y_{AB}}{\partial\alpha} \tag{28}$$

$$\frac{\gamma_1}{\gamma_1+\gamma_2}Y_{AB} = \frac{(1-\beta)}{m}\frac{\partial Y_{AB}}{\partial\beta} \tag{29}$$

Dividing equation 29 by equation 28 produces the first result. Adding the two equations produces the second. ■

Corollary 20 If Y_{AB} is homogeneous of degree one in α and β, then at every interior Cournot equilibrium we will have $Y_{AB} = \frac{1}{m+1}\left(\frac{\partial Y_{AB}}{\partial \alpha} + \frac{\partial Y_{AB}}{\partial \beta}\right)$

Proof. Condition 27 implies that $mY_{AB} = \frac{\partial Y_{AB}}{\partial \alpha} + \frac{\partial Y_{AB}}{\partial \beta} - \left(\alpha\frac{\partial Y_{AB}}{\partial \alpha} + \beta\frac{\partial Y_{AB}}{\partial \beta}\right)$. Euler's theorem implies that $\alpha\frac{\partial Y_{AB}}{\partial \alpha} + \beta\frac{\partial Y_{AB}}{\partial \beta} = Y_{AB}$. Rearranging produces the result. ■

Corollary 21 Let $\left(\alpha; \beta\right)$ be any Cournot equilibrium in the interior of the unit square. Then with our functional choices

$$\frac{(1-\alpha)^{m+1} f^m k^{m+1}}{(1-\beta)^{m+1}} = \frac{1}{p} \tag{30}$$

$$pk\left[\alpha(m+1) - 1\right] + \beta(m+1) - 1 = 0 \tag{31}$$

Proposition 22 For any Cournot equilibrium $\left(\alpha; \beta\right)$

$$\alpha \lesseqgtr \frac{1}{m+1} \iff \beta \gtreqless \frac{1}{m+1}$$

Proof. An interior equilibrium has to satisfy equation 31. Equality can hold only if $[\alpha(m+1) - 1]$ and $[\beta(m+1) - 1]$ have opposite signs. In the case of corner equilibria we have immediately $\alpha = 0 \Rightarrow \beta > \frac{1}{m+1}$ and $\beta = 0 \Rightarrow \alpha > \frac{1}{m+1}$. ■

Remark 23 At any interior Cournot equilibrium we will have

$$g_A = \frac{f^{\frac{m}{m+1}}}{p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}},\ g_B = \frac{p^{\frac{m}{m+1}}}{p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}}$$

and

$$Y_1 = \frac{f^{\frac{m}{m+1}}}{p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}}\frac{c(pk+1)W}{m+1},\ Y_2 = \frac{p^{\frac{m}{m+1}}}{p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}}\frac{c(pk+1)W}{m+1}$$

Proof. We can rewrite equation 30 as $\frac{(1-\alpha)^{m+1} f^{m+1} k^{m+1}}{(1-\beta)^{m+1}} = \frac{f}{p}$, hence $\frac{(1-\alpha)^m f^m k^m}{(1-\beta)^m} = \left(\frac{f}{p}\right)^{\frac{m}{m+1}}$. The left hand side of this is just $\frac{g_A}{g_B}$. Then exploiting the fact that $g_A + g_B = 1$, we can solve for g_A and g_B. To get the expressions for Y_1 and Y_2 we note that $Y_1 = g_A Y_{AB}$ and $Y_2 = g_B Y_{AB}$ By corollary 20 we have that $Y_{AB} = \frac{c(pk+1)W}{m+1}$ with our specification. The result follows. ■

Theorem 24 Let (α, δ) be any Cournot equilibrium, then

$$\alpha = F_1(k; p; f; m)$$
$$\delta = F_2(k; p; f; m)$$

Furthermore if (α, δ) is an interior equilibrium then

$$\alpha = 1 - \frac{m}{m+1}\left(\frac{p^{\frac{m}{m+1}}}{p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}}\right)\frac{pk+1}{pk} \tag{32a}$$

$$\delta = 1 - \frac{m}{m+1}\left(\frac{f^{\frac{m}{m+1}}}{p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}}\right)(pk+1) \tag{32b}$$

while if $(0; \beta^*)$ is a corner equilibrium, then β^* is implicitly defined by

$$(1 - \beta^*)^{m+1} = f^m k^m (\beta^*(m+1) - 1)$$

and if $(\alpha^*; 0)$ is a corner equilibrium, then α^* is implicitly defined by

$$(1 - \alpha^*)^{m+1} f^m k^m = \alpha^*(m+1) - 1$$

Furthermore (α, δ) will be an interior Cournot equilibrium if, and only if[13],

$$\frac{m}{p\left[1 + (m+1)\left(\frac{f}{p}\right)^{\frac{m}{m+1}}\right]} \le k \le \frac{1 + (m+1)\left(\frac{p}{f}\right)^{\frac{m}{m+1}}}{mp} \tag{33}$$

$(\alpha^*; 0)$ will be a corner Cournot equilibrium if, and only if, $k \ge \frac{1+(m+1)\left(\frac{p}{f}\right)^{\frac{m}{m+1}}}{mp}$ and $(0; \beta^*)$ will be a corner Cournot equilibrium if, and only if, $k \le \frac{m}{p\left[1+(m+1)\left(\frac{f}{p}\right)^{\frac{m}{m+1}}\right]}$

Proof. Note that the theorem states that the Cournot solution will be independent of c and W.

[13] Note that $\frac{m}{p\left[1+(m+1)\left(\frac{f}{p}\right)^{\frac{m}{m+1}}\right]} < \frac{1+(m+1)\left(\frac{p}{f}\right)^{\frac{m}{m+1}}}{mp}$ for all choices of the parameters.

We will derive the conditions for the interior equilibrium first. Equation 30 in corollary 21 can be rewritten as $\frac{(1-\alpha)^{m+1} f^{m+1} k^{m+1}}{(1-\beta)^{m+1}} = \frac{f}{p}$, i.e.

$$\frac{(1-\alpha) f k}{(1-\beta)} = \left(\frac{f}{p}\right)^{\frac{1}{m+1}} \tag{34}$$

so $(1-\alpha)k = \left(\frac{f}{p}\right)^{\frac{1}{m+1}} (1-\beta) f^{-1}$. From equation 31 we get $k = \frac{1-\beta(m+1)}{p(\alpha(m+1)-1)}$. Substituting this in, and cross-multiplying we get

$$(1-\alpha)(1-\beta(m+1)) = \left(\frac{p}{f}\right)^{\frac{m}{m+1}} (1-\beta)(\alpha(m+1)-1)$$

Collecting up terms in α and simplifying we get

$$1-\alpha = \frac{\left(\frac{p}{f}\right)^{\frac{m}{m+1}} (1-\beta) m}{\left(\frac{p}{f}\right)^{\frac{m}{m+1}} (m+1)(1-\beta) - \beta(m+1) + 1}$$

We substitute this expression back into the equation 34 and solve for β. This gives us equation 32b. With this expression we substitute back into the equation above, to obtain the equilibrium value of α, given in equation 32a.

The solutions given in equations 32a and 32b represent a legitimate equilibrium only if both formulae evaluate to non-negative quantities (they are guaranteed to produce values less than one). Imposing the conditions $\alpha \ge 0$ and $\beta \ge 0$ and simplifying the respective equations yields the conditions in (33) above.

To see what happens at the corner, let us assume that $(\alpha^*; 0)$ is a corner Cournot equilibrium. In this case α^* solves $\frac{\partial Y_1}{\partial \alpha} = 0$ at $\beta = 0$. Furthermore we have that $\frac{\partial Y_2}{\partial \beta} \le 0$ at $(\alpha^*; 0)$. By proposition 4 the following two conditions must be satisfied at the point $(\alpha^*; 0)$:

$$\frac{1}{(1-\alpha)^m f^m k^m + 1}(p\alpha k) = \frac{(1-\alpha) pk}{m} \tag{35a}$$

$$\frac{(1-\alpha)^m f^m k^m}{(1-\alpha)^m f^m k^m + 1}(p\alpha k) \ge \frac{1}{m} \tag{35b}$$

(Compare with equations 23 and 24 with $\beta = 0$). The first equation produces the condition

$$(1-\alpha)^{m+1} f^m k^m = \alpha(m+1) - 1 \tag{36}$$

(as in Theorem 17 when read with Definition 16). When we divide each side of inequality 35b by the corresponding two sides of equation 35a we get

$$(1-\alpha)^m f^m k^m \geq \frac{1}{(1-\alpha)pk}$$

i.e. $(1-\alpha)^{m+1} f^{m+1} k^{m+1} > \frac{f}{p}$, i.e.

$$(1-\alpha)^m f^m k^m \geq \left(\frac{f}{p}\right)^{\frac{m}{m+1}} \tag{37}$$

From equation 36 we get that $(1-\alpha)^m f^m k^m = \frac{\alpha(m+1)-1}{(1-\alpha)}$. Substituting this into the left-hand side of the inequality above and simplifying we get

$$\alpha \geq \frac{1+\left(\frac{p}{f}\right)^{\frac{m}{m+1}}}{(m+1)\left(\frac{p}{f}\right)^{\frac{m}{m+1}}+1}$$

From this it follows that

$$\frac{1}{1-\alpha} \geq \frac{(m+1)\left(\frac{p}{f}\right)^{\frac{m}{m+1}}+1}{m\left(\frac{p}{f}\right)^{\frac{m}{m+1}}}$$

We can rewrite inequality 37 as

$$k \geq \frac{1}{(1-\alpha)f}\left(\frac{f}{p}\right)^{\frac{1}{m+1}}$$

Substituting in for $\frac{1}{1-\alpha}$ and simplifying yields

$$k \geq \frac{(m+1)\left(\frac{p}{f}\right)^{\frac{m}{m+1}}+1}{mp}$$

We have therefore shown that $(\alpha^*, 0)$ is a corner Cournot equilibrium only if $k \geq \frac{(m+1)\left(\frac{p}{f}\right)^{\frac{m}{m+1}}+1}{mp}$

The conditions for the other corner equilibrium follow similarly.

Since there is always guaranteed to be a Cournot equilibrium; since the three cases exhaust all possibilities; and because these equilibria are uniquely defined by k, p, f and m, the opposite implications follow. ■

Remark 25 At any corner Cournot equilibrium $(\alpha^*; 0)$ we will have

$$g_A = \frac{\alpha(m+1) - 1}{\alpha m}, \quad g_B = \frac{(1-\alpha)}{\alpha m}$$

and

$$Y_1 = \frac{cpkW\left[\alpha(m+1) - 1\right]}{m}, \quad Y_2 = \frac{(1-\alpha)\, cpkW}{m}$$

At any corner Cournot equilibrium $(0; \beta^*)$ we will have

$$g_A = \frac{(1-\beta)}{\beta m}, \quad g_B = \frac{\beta(m+1) - 1}{\beta m}$$

and

$$Y_1 = \frac{(1-\beta)\, cW}{m}, \quad Y_2 = \frac{\left[\beta(m+1) - 1\right] cW}{m}$$

Proof. Consider a corner equilibrium $(\alpha; 0)$. In order for α to be an optimum, it must satisfy equation 23. Substituting in $\beta = 0$ and simplifying, we get.

$$\frac{p\alpha k}{(1-\alpha)^m f^m k^m + 1} = \frac{(1-\alpha)\, pk}{m}$$

The left hand side of this expression is $\frac{Y_2}{cW}$. We obtain Y_1 from the fact that $Y_1 + Y_2 = Y_{AB}$ and total production at this corner is simply $cp\alpha kW$. Furthermore the expression above simplifies to:

$$\frac{1}{(1-\alpha)^m f^m k^m + 1} = \frac{(1-\alpha)}{\alpha m}$$

The expression on the left is g_B. From the fact that $g_A = 1 - g_B$ we deduce that $g_A = \frac{\alpha(m+1)-1}{\alpha m}$. The results for the corner $(0; \beta)$ follow in like fashion, by noting that the optimal β has to satisfy 24 with $\alpha = 0$. ■

Definition 26 Let γ be such that $\ln \gamma = \frac{1}{\gamma} + 1$

We have $\gamma \approx 3{:}591\,121\,477$. Note that the function $\ln y - 1 - \frac{1}{y}$ is strictly increasing. So for $y < \gamma$ we will have $\ln y - 1 - \frac{1}{y} < 0$ and for $y > \gamma$ we will have $\ln y - 1 - \frac{1}{y} > 0$

Proposition 27 Let (α, δ) be any interior Cournot equilibrium. Then

$$
\begin{aligned}
\frac{\partial \alpha}{\partial k} &> 0, \ \frac{\partial \delta}{\partial k} < 0 \\
\frac{\partial \alpha}{\partial p} &> 0, \ \frac{\partial \delta}{\partial p} < 0 \\
\frac{\partial \alpha}{\partial f} &> 0, \ \frac{\partial \delta}{\partial f} < 0 \\
\frac{\partial \alpha}{\partial m} &\to 0 \ \text{ as } \left(\frac{f}{p}\right)^{\frac{m}{m+1}} \to \infty, \ \frac{\partial \delta}{\partial m} \to 0 \ \text{ as } \left(\frac{p}{f}\right)^{\frac{m}{m+1}} \to \infty
\end{aligned}
$$

Proof. The results follow by differentiating the expressions for α and δ. The comparative statics with respect to k are given by

$$\frac{\partial \alpha}{\partial k} = \frac{m}{m+1} \frac{p^{\frac{m}{m+1}}}{p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}} \frac{1}{pk^2}$$

and

$$\frac{\partial \delta}{\partial k} = -\frac{m}{m+1} \left(\frac{f^{\frac{m}{m+1}}}{p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}} \right) p$$

The derivative with respect to p is slightly more tricky. After some algebraic manipulation, it turns out that

$$\frac{\partial \alpha}{\partial p} = \frac{m}{m+1} \frac{p^{\frac{m}{m+1}}}{p^2 \left(p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}} \right)} \frac{\delta}{k}$$

This will be positive provided that $\delta > 0$. Once β reaches the boundary we see that $\frac{\partial \alpha}{\partial p} = 0$. Indeed it is obvious that p does not feature in the equations for the corner equilibria.

Similarly

$$\frac{\partial \delta}{\partial p} = -\frac{m}{m+1} \left(\frac{f^{\frac{m}{m+1}}}{p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}} \right) \alpha k$$

The derivatives with respect to f are straightforward and given by

$$\frac{\partial \alpha}{\partial f} = \frac{m^2}{(m+1)^2} \frac{p^{\frac{m}{m+1}}}{f^{\frac{1}{m+1}} \left(p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}} \right)^2} \frac{(pk+1)}{pk}$$

and

$$\frac{\partial \beta}{\partial f} = -\frac{m^2}{(m+1)^2}\frac{p^{\frac{m}{m+1}}}{f^{\frac{1}{m+1}}\left(p^{\frac{m}{m+1}}+f^{\frac{m}{m+1}}\right)^2}(pk+1)$$

In order to calculate the comparative statics with respect to m, it is in fact easier to calculate them with respect to the variable $\mu = \frac{m}{m+1}$. Since $\frac{d\mu}{dm} > 0$ the sign of $\frac{\partial \alpha}{\partial \mu}$ will correspond to the sign of $\frac{\partial \alpha}{\partial m}$. It is also more convenient to rewrite α as

$$\alpha = 1 - \frac{1}{1+\left(\frac{f}{p}\right)^{\mu}}\frac{pk+1}{pk}$$

Then

$$\frac{\partial \alpha}{\partial \mu} = -\frac{1+\left(\frac{f}{p}\right)^{\mu} - \mu\left(\frac{f}{p}\right)^{\mu}\ln\left(\frac{f}{p}\right)}{\left[1+\left(\frac{f}{p}\right)^{\mu}\right]^2}\left(\frac{pk+1}{pk}\right)$$

The sign of this expression depends only on the sign of $1+\left(\frac{f}{p}\right)^{\mu} - \mu\left(\frac{f}{p}\right)^{\mu}\ln\left(\frac{f}{p}\right)$. We can rewrite this as $-\left[\left(\frac{f}{p}\right)^{\mu}\ln\left(\frac{f}{p}\right)^{\mu} - \left(\frac{f}{p}\right)^{\mu} - 1\right]$, i.e. as $-\left(\frac{f}{p}\right)^{\mu}\left[\ln\left(\frac{f}{p}\right)^{\mu} - 1 - \frac{1}{\left(\frac{f}{p}\right)^{\mu}}\right]$. The expression in square brackets is of the form $\ln y - 1 - \frac{1}{y}$. Consequently $\frac{\partial \alpha}{\partial \mu} \lesseqgtr 0$ as $\left(\frac{f}{p}\right)^{\mu} \lesseqgtr 3$. The result follows. Similarly $\beta = 1 - \frac{1}{\left(\frac{p}{f}\right)^{\mu}+1}(pk+1)$. We find that $\frac{\partial \beta}{\partial \mu} \lesseqgtr 0$ as $\left(\frac{p}{f}\right)^{\mu} \lesseqgtr 3$. ■

Proposition 28 Let $(\alpha^*; 0)$ be a Cournot equilibrium. Then

$$\frac{\partial \alpha^*}{\partial k} > 0,\ \frac{\partial \alpha^*}{\partial f} > 0,\ \frac{\partial \alpha^*}{\partial m} \lesseqgtr 0 \text{ as } (1-\alpha^*)^m f^m k^m \lesseqgtr 3$$

Similarly, if $(0; \beta^*)$ is a Cournot equilibrium, then

$$\frac{\partial \beta^*}{\partial k} < 0,\ \frac{\partial \beta^*}{\partial f} < 0,\ \frac{\partial \beta^*}{\partial m} \lesseqgtr 0 \text{ as } \frac{(1-\beta^*)^m}{f^m k^m} \lesseqgtr 3$$

Proof. The proof proceeds by implicit differentiation. Let $q(\alpha) = (1-\alpha)^{m+1} f^m k^m - \alpha(m+1) + 1$. Then α^* is defined implicitly by $q(\alpha) \equiv 0$. $\frac{\partial \alpha^*}{\partial k} = -\frac{q_k}{q_\alpha}$, provided that $q_\alpha \neq 0$. We have $q_\alpha = -(m+1)(1-\alpha)^m f^m k^m - (m+1) < 0$. $q_k = m(1-\alpha)^{m+1} f^m k^{m-1} > 0$, hence

$$\frac{\partial \alpha^*}{\partial k} = \frac{m(1-\alpha)^{m+1} f^m k^{m-1}}{(m+1)\left[(1-\alpha)^m f^m k^m + 1\right]}$$

which is positive. We can rewrite this expression utilising the fact that $(1-\alpha)^{m+1} f^m k^m = \alpha(m+1) - 1$. We get

$$\frac{\partial \alpha^*}{\partial k} = \frac{(1-\alpha)\left[\alpha(m+1) - 1\right]}{(m+1)\alpha k} \tag{38}$$

Also $q_f = (1-\alpha)^{m+1} f^{m-1} k^m > 0$, i.e. $\frac{\partial \alpha^*}{\partial f} > 0$.

In the case of m we have $q_m = (1-\alpha)^{m+1} f^m k^m \ln\left[(1-\alpha) f k\right] - \alpha$. It helps to rewrite this. Let $y = (1-\alpha)^m f^m k^m$, then $q_m = (1-\alpha) y \ln y^{\frac{1}{m}} - \alpha$, i.e. $q_m = \frac{1-\alpha}{m}\left[y \ln y - \frac{\alpha m}{1-\alpha}\right]$. This expression has to be evaluated at a solution to $q(\alpha) = 0$. At such a solution we have $\alpha m = (1-\alpha)^{m+1} f^m k^m + (1-\alpha)$. Substituting this into the expression for q_m we get $q_m = \frac{1-\alpha}{m}\left[y \ln y - y + 1\right]$, i.e. $q_m = \frac{1-\alpha}{m} y \left[\ln y - 1 - \frac{1}{y}\right]$. So $q_m \gtreqless 0$ as $(1-\alpha^*)^m f^m k^m \gtreqless \zeta$. The results for β^* follow symmetrically. ■

Theorem 29 Let $(\alpha; \beta)$ be any Cournot equilibrium. Then

$$\frac{\partial \alpha}{\partial k} \geq 0, \quad \frac{\partial \beta}{\partial k} \leq 0$$

$$\frac{\partial \alpha}{\partial p} \geq 0, \quad \frac{\partial \beta}{\partial p} \leq 0$$

$$\frac{\partial \alpha}{\partial f} \geq 0, \quad \frac{\partial \beta}{\partial f} \leq 0$$

$$\frac{\partial \alpha}{\partial m} \gtreqless 0 \text{ as } \frac{g_A}{g_B} \gtreqless \zeta, \quad \frac{\partial \beta}{\partial m} \gtreqless 0 \text{ as } \frac{g_B}{g_A} \gtreqless \zeta$$

Proof. This proof merely summarises the results of propositions 27 and 28. Because of the possibility of being at a corner, the inequalities have been weakened, to allow for the fact that when $\beta = 0$, we certainly do not have $\frac{\partial \beta}{\partial k} < 0$. Furthermore at the corner both $\frac{\partial \alpha}{\partial p} = 0$ and $\frac{\partial \beta}{\partial p} = 0$. We have also characterised the comparative statics for $\underline{m}$ somewhat differently. We have utilised the fact that for interior solutions $\left(\frac{f}{p}\right)^{\frac{m}{m+1}} = \frac{g_A}{g_B}$, while if $\beta = 0$, then $\frac{g_A}{g_B} = (1 - \alpha^*)^m f^m k^m$. ■

A.4 Intertemporal equilibria

We assume, as in the text (equation 6) that

$$W_{A;t+1} = Y_{1;t} \qquad W_{B;t+1} = Y_{2;t}$$

and that an intertemporal equilibrium is defined by $k_{t+1} = k_t$. We will show in the next theorem that under certain circumstances this requirement is sufficient to define $\underline{k}$ uniquely as a function of the other parameters.

Theorem 30 Let $(\alpha^*; \beta^*; k^*)$ be any intertemporal equilibrium, then if $m \leq 1$ the equilibrium values will be uniquely defined by $(p; f; m)$, i.e.

$$\begin{aligned} k^* &= G_1(p; f; m) \\ \alpha^* &= G_2(p; f; m) \\ \beta^* &= G_3(p; f; m) \end{aligned}$$

In particular, if $(\alpha^*; \beta^*; k^*)$ is an interior intertemporal Cournot equilibrium, then

$$k^* = \left(\frac{f}{p}\right)^{\frac{m}{m+1}} \qquad (39a)$$

$$\alpha^* = 1 - \frac{m}{m+1}\left(\frac{p^{\frac{m}{m+1}}}{p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}}\right)\left(\frac{p^{\frac{1}{m+1}} f^{\frac{m}{m+1}} + 1}{p^{\frac{1}{m+1}} f^{\frac{m}{m+1}}}\right) \qquad (39b)$$

$$\beta^* = 1 - \frac{m}{m+1}\left(\frac{f^{\frac{m}{m+1}}}{p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}}\right)\left(p^{\frac{1}{m+1}} f^{\frac{m}{m+1}} + 1\right) \qquad (39c)$$

If $(\alpha^*; 0; k^*)$ is a corner equilibrium, then k^* and α^* are defined through the equations

$$f = k^{*\frac{1-m}{m}} \left(\frac{m + k^* + 1}{m}\right) \qquad (40a)$$

$$\alpha^* = \frac{k^* + 1}{m + k^* + 1} \qquad (40b)$$

If $\left(0; \hat{\beta}; \hat{k}\right)$ is a corner equilibrium, then $\hat{k}$ and $\hat{\beta}$ are defined through the equations

$$f = \frac{\hat{k}^{\frac{1}{m}} m}{\hat{k} m + \hat{k} + 1} \tag{41a}$$

$$\hat{\beta} = \frac{\hat{k} + 1}{\hat{k} m + \hat{k} + 1} \tag{41b}$$

$\left(\hat{\alpha}; \hat{\beta}; \hat{k}\right)$ will be an interior equilibrium if and only if

$$p^{\frac{m}{m+1}} \left\{ \frac{-1 + \sqrt{1 + \frac{4m(m+1)}{p}}}{2(m+1)} \right\} \leq f^{\frac{m}{m+1}} \leq p^{\frac{m}{m+1}} \left(\frac{1 + \sqrt{1 + 4m(m+1)p}}{2mp} \right) \tag{42}$$

If $m > 1$ then it is possible for there to be multiple equilibria, i.e. the same values of $p; m;$ and f might combine with different values of k to yield intertemporal equilibria. The equations governing each equilibrium will be given by the equations above.

Proof. By the condition for intertemporal equilibrium we have that $k = \frac{(1-\alpha)^m f^m k^m}{(1-\beta)^m}$. However by equation 34 we know that at an interior Cournot equilibrium we must have

$$\frac{(1-\hat{\alpha})^m f^m k^m}{\left(1-\hat{\beta}\right)^m} = \left(\frac{f}{p}\right)^{\frac{m}{m+1}}$$

The first result therefore follows. We substitute this value of k into the equilibrium values of α and β given in equations 32a and 32b to get the other results.

This solution will, however, define a legitimate equilibrium only if $\hat{\alpha} \geq 0$ and $\hat{\beta} \geq 0$. It is obvious that both will be less than one. The condition $\hat{\beta} \geq 0$ will be true only if $\left(mp^{\frac{1}{m+1}} f^{\frac{m}{m+1}}\right)^2 - f^{\frac{m}{m+1}} - (m+1)p^{\frac{m}{m+1}} \leq 0$. This is a quadratic in $f^{\frac{m}{m+1}}$ which will always have two real roots. Applying the quadratic formula and noting that $f^{\frac{m}{m+1}}$ would have to be smaller than the positive root, we get the inequality

$$f^{\frac{m}{m+1}} \leq \frac{1 + \sqrt{1 + 4(m+1)mp}}{2mp^{\frac{1}{m+1}}}$$

The other inequality is found by considering the condition $\tilde{\alpha} \geq 0$ (or by applying the symmetry rules).

To see what happens at the corner, we note that if $\tilde{\beta} = 0$, then as in Theorem 24, the optimal α has to satisfy the condition $\alpha(m+1) - 1 = (1-\alpha)^{m+1} f^m k^m$. We note also that the condition for intertemporal equilibrium now requires that $k = \frac{(1-\alpha)^m f^m k^m}{1}$. Substituting $(1-\alpha)^m f^m k^m = k$ into the right hand side of the previous equation, we get $\alpha(m+1) - 1 = (1-\alpha)k$ and the equilibrium condition on $\tilde{\alpha}$ follows. Substituting $\alpha = \frac{\tilde{k}+1}{m+\tilde{k}+1}$ into $\alpha(m+1) - 1 = (1-\alpha)^{m+1} f^m k^m$ and simplifying gives the equilibrium condition for $\tilde{k}$.

The conditions for the other corner can be derived equivalently (or by symmetry).

To show uniqueness, we assume that $k_0 = \left(\frac{f}{p}\right)^{\frac{m}{m+1}}$ is such that $k_0 < \frac{(m+1)\left(\frac{p}{f}\right)^{\frac{m}{m+1}}+1}{mp}$. By Theorem 24 it follows that the combination of parameters $(k_0; p; m; f)$ defines either an interior equilibrium or a corner equilibrium with $\alpha = 0$. Assume now that there is some $k = k_1$ also such that $(k_1; p; m; f)$ defines a corner equilibrium with $\tilde{\beta} = 0$. It follows from Theorem 24 that $k_1 \geq \frac{(m+1)\left(\frac{p}{f}\right)^{\frac{m}{m+1}}+1}{mp}$, i.e. $k_1 > \left(\frac{f}{p}\right)^{\frac{m}{m+1}}$. Now k_1 has to satisfy equation 40a, i.e. $f = k_1^{\frac{1-m}{m}} \left[\frac{m+k_1+1}{m}\right]$. Since we have assumed that $m \leq 1$, $k_1^{\frac{1-m}{m}} \geq \left(\frac{f}{p}\right)^{\frac{1-m}{m+1}}$, so $f > \left(\frac{f}{p}\right)^{\frac{1-m}{m+1}} \left(\frac{m+\left(\frac{f}{p}\right)^{\frac{m}{m+1}}+1}{m}\right)$. Simplifying we get $mpf^{\frac{2m}{m+1}} > (m+1)p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}$. This, however, contradicts the assumption that $\left(\frac{f}{p}\right)^{\frac{m}{m+1}} < \frac{(m+1)\left(\frac{p}{f}\right)^{\frac{m}{m+1}}+1}{mp}$. It follows that there can be no k_1 such that $F_2(k_1; p; m; f) = 0$.

Similarly we show that if $k_0 = \left(\frac{f}{p}\right)^{\frac{m}{m+1}}$ is such that $k_0 > \frac{m}{p\left[1+(m+1)\left(\frac{f}{p}\right)^{\frac{m}{m+1}}\right]}$, then there can be no corner solution with $\alpha = 0$. Putting the two results together it follows that if

$$\frac{m}{p\left[1+(m+1)\left(\frac{f}{p}\right)^{\frac{m}{m+1}}\right]} < \left(\frac{f}{p}\right)^{\frac{m}{m+1}} < \frac{(m+1)\left(\frac{p}{f}\right)^{\frac{m}{m+1}}+1}{mp} \qquad (43)$$

then we cannot have any corner solutions. Hence we can only have an interior solution. This solution will be given by equations 39a, 39b and 39c. The uniqueness of the corner solutions follows from the fact that if inequality 43 does not hold we can either have $\left(\frac{f}{p}\right)^{\frac{m}{m+1}} \geq \frac{(m+1)\left(\frac{p}{f}\right)^{\frac{m}{m+1}}+1}{mp}$, in which case $\beta = 0$ or $\frac{m}{p\left[1+(m+1)\left(\frac{f}{p}\right)^{\frac{m}{m+1}}\right]} \geq \left(\frac{f}{p}\right)^{\frac{m}{m+1}}$, in which case $\alpha = 0$, but not both.

Note that condition 43 can be rearranged to yield condition 42.

To show that this result does not hold if $m > 1$, it is easiest to do so with a counterexample. Let $m = 10$, $p = 0.04$ and $f = 0.2643743365$. Then there are the following three intertemporal equilibria:

1. $k = 10$, $\alpha = .5238095238$, $\beta = 0$.
2. $k = 5.566719865$, $\alpha = 0.2398342553$, $\beta = 0.057.74810407$
3. $k = 3.843310204$, $\alpha = 0$, $\beta = 0.1119157054$

■

Corollary 31 If $f > 1$ and $p > 1$ and $m < 1$, then we cannot have a corner solution in which $\alpha = 0$. If $f < 1$ and $p < 1$ and $m < 1$, then we cannot have a corner solution in which $\beta = 0$.

Proof. If $f > 1$ and $p > 1$, then $\frac{m}{p\left[1+(m+1)\left(\frac{f}{p}\right)^{\frac{m}{m+1}}\right]} < \left(\frac{f}{p}\right)^{\frac{m}{m+1}}$. Since equilibria are unique if $m < 1$, it follows that there cannot be corner equilibrium in which $\alpha = 0$. If $f < 1$ and $p < 1$, then $\left(\frac{f}{p}\right)^{\frac{m}{m+1}} < \frac{(m+1)\left(\frac{p}{f}\right)^{\frac{m}{m+1}}+1}{mp}$. ■

Remark 32 The conditions for the corner equilibria can be presented equivalently in terms of α and β (rather than k):

If $\left(\hat{\alpha}; 0; \hat{\beta}\right)$ is a corner equilibrium, then $\hat{\alpha}$ and $\hat{\beta}$ are defined through the equations

$$[\hat{\alpha}(m+1) - 1]^{1-m} = (1 - \hat{\alpha}) f^m$$
$$\hat{\beta} = \frac{\hat{\alpha}(m+1) - 1}{1 - \hat{\alpha}}$$

If $\left(0;\hat{\beta};\hat{k}\right)$ is a corner equilibrium, then $\hat{\beta}$ and $\hat{k}$ are defined through the equations

$$\left[\hat{\beta}(m+1)-1\right]^{1-m} f^{m} = \left(1-\hat{\beta}\right)$$

$$\hat{k} = \frac{1-\hat{\beta}}{\hat{\beta}(m+1)-1}$$

Proposition 33 An interior Cournot equilibrium will always be stable. A corner equilibrium $(\alpha;0;k)$ or $(0;\beta;k)$ will be stable if $\alpha(\beta) > \frac{m}{m+1}$ and will be unstable if $\alpha(\beta) < \frac{m}{m+1}$.

Proof. Let $D(k_t) = k_t - \frac{Y_1(k_t)}{Y_2(k_t)}$. Note that D is a continuous function of k, since the locus of Cournot equilibria is continuous. By condition 6 we have $k_{t+1} = \frac{Y_1(k_t)}{Y_2(k_t)}$, so clearly $D(k) = 0$ if, and only if, k is an intertemporal equilibrium. Now consider first an interior intertemporal equilibrium at $\hat{k}$. Now consider the set of all ks which meet condition 33. This will always be a non-empty set. By remark 23 every one of these k's will give $\frac{Y_1(k)}{Y_2(k)} = \left(\frac{f}{p}\right)^{\frac{m}{m+1}} = \hat{k}$. In a neighbourhood of $\hat{k}$ therefore we have $D(k) > 0$ if $k > \hat{k}$ and $D(k) < 0$ if $k < \hat{k}$. This, however, establishes the stability of $\hat{k}$.[14]

Now consider a solution to the equation $D(k) = 0$ at a corner $(\alpha^*;0;k^*)$. Now k^* must be such that it the condition $k \geq \frac{1+(m+1)\left(\frac{p}{f}\right)^{\frac{m}{m+1}}}{mp}$ must hold with a strict inequality (otherwise $\frac{Y_1}{Y_2} = \left(\frac{f}{p}\right)^{\frac{m}{m+1}}$ and k^* would not be an intertemporal equilibrium). This means that there is a neighbourhood around k^* in which there are only corner solutions with $\beta = 0$. In particular every $k > k^*$ will yield a corner solution. Now by remark 25 at every one of these k's we will have $\frac{Y_1}{Y_2} = \frac{\alpha(m+1)-1}{1-\alpha}$, i.e. $D(k) = k - \frac{\alpha(m+1)-1}{1-\alpha}$. Consequently $D'(k) = 1 - \frac{m}{(1-\alpha)^2}\frac{\partial\alpha}{\partial k}$. Substituting in $\frac{\partial\alpha}{\partial k}$ from equation 38 we get

$$D'(k) = 1 - \frac{m}{(1-\alpha)^2}\frac{(1-\alpha)\left[\alpha(m+1)-1\right]}{(m+1)\alpha k}$$

i.e.

$$D'(k) = 1 - \frac{m\left[\alpha(m+1)-1\right]}{(m+1)(1-\alpha)\alpha k}$$

[14] It also establishes that there can only ever be one interior intertemporal equilibrium.

when we evaluate this at the equilibrium $(\alpha^*; 0; k^*)$ we note that by remark 32 we have $k^* = \frac{\alpha^*(m+1)-1}{1-\alpha^*}$, i.e. $D'(k^*) = 1 - \frac{m}{(m+1)\alpha^*}$. We will have $D'(k^*) > 0$ only if $\alpha^* > \frac{m}{m+1}$. In that case there will be a neighbourhood around k^* on which $k_t > k_{t+1}$ if $k_t > k^*$ and $k_t < k_{t+1}$ if $k_t < k^*$. Furthermore if $k_t > k^*$, then by proposition 28 $\alpha(k_t) > \alpha(k^*)$, i.e. $\frac{\alpha_t(m+1)-1}{1-\alpha_t} > \frac{\alpha^*(m+1)-1}{1-\alpha^*} = k^*$. Hence $k_t > k_{t+1} > k^*$. This establishes local stability.

If $\alpha^* < \frac{m}{m+1}$, then we will have $D'(k^*) < 0$ in a neighbourhood of k^* and then points in this neighbourhood will diverge from k^*.

The results for the other corner follow symmetrically. ■

Proposition 34 If $\left(\alpha^*; \beta^*; k^*\right)$ is an interior intertemporal Cournot equilibrium, then

$$\frac{\partial k}{\partial p} < 0,\ \frac{\partial k}{\partial f} > 0,\ \frac{\partial k}{\partial m} \gtrless 0 \text{ as } \left(\frac{f}{p}\right) \gtrless 1$$

$$\frac{\partial \alpha}{\partial p} \lesseqgtr 0 \text{ as } \beta \lesseqgtr \frac{m}{m+1},\ \frac{\partial \alpha}{\partial f} > 0,\ \frac{\partial \alpha}{\partial m} < 0 \text{ if } f < p \text{ and } \frac{\partial \alpha}{\partial m} > 0 \text{ if } f > \left(\frac{m+1}{m}\right)p$$

$$\frac{\partial \beta}{\partial p} \lesseqgtr 0 \text{ as } \alpha \gtreqless \frac{m}{m+1},\ \frac{\partial \beta}{\partial f} < 0,\ \frac{\partial \beta}{\partial m} < 0 \text{ if } p < f \text{ and } \frac{\partial \beta}{\partial m} > 0 \text{ if } p > \left(\frac{m+1}{m}\right)f$$

Proof. To get the comparative statics, for k is straightforward. To get the others, we could just differentiate the expressions above. A more instructive and easier route is to use the partial derivatives already obtained. So totally differentiating (and abusing notation somewhat) we get

$$\frac{\partial \alpha}{\partial p} = \frac{\partial \alpha}{\partial p} + \frac{\partial \alpha}{\partial k}\frac{\partial k}{\partial p}$$

The virtue of thinking about the processes in this way is that it is explicit that there are two types of changes which will influence the optimal choice of α: firstly there is the direct effect (which we know is positive); secondly, however, there is a wealth effect which might offset the first effect. We know that the optimal α increases with p. The increasing p, however, also has the effect of decreasing the equilibrium wealth of player A. Substituting in for all of the expressions, and simplifying, we get

$$\frac{\partial \alpha}{\partial p} = \frac{m}{m+1}\frac{p^{\frac{2m}{m+1}}\left(\beta - \frac{m}{m+1}\right)}{p^2\left(p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}\right)f^{\frac{m}{m+1}}}$$

Similarly we get

$$\frac{\partial \epsilon}{\partial p} = -\frac{m}{m+1}\frac{f^{\frac{m}{m+1}}k\left[\epsilon - \frac{m}{m+1}\right]}{p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}}$$

The partial derivatives with respect to f are even easier, since $\frac{\partial \epsilon}{\partial f} = \frac{\partial \alpha}{\partial f} + \frac{\partial \alpha}{\partial k}\frac{\partial k}{\partial f}$ and every one of these terms is positive, while $\frac{\partial \bar{\beta}}{\partial f}$ and $\frac{\partial \bar{\beta}}{\partial k}$ are both negative.

We have $\frac{\partial \alpha}{\partial m} < 0$ if $\frac{g_A}{g_B} < \zeta$ but $k = \frac{g_A}{g_B}$, so $\frac{\partial \alpha}{\partial m} < 0$ if $k < \zeta$. Furthermore $\frac{\partial k}{\partial m} < 0$ if $\left(\frac{f}{p}\right) < 1$. We can sign $\frac{\partial \epsilon}{\partial m}$ if $f < p$, since in that case $k = \left(\frac{f}{p}\right)^{\frac{m}{m+1}} < 1$, i.e. $\frac{\partial \alpha}{\partial m} < 0$. If $\left(\frac{f}{p}\right)^{\frac{m}{m+1}} > \zeta$, then both $\frac{\partial \alpha}{\partial m}$ and $\frac{\partial k}{\partial m}$ are positive. ■

Proposition 35 If $(\epsilon; 0; \rho)$ is a corner equilibrium, then

$$\frac{\partial \rho}{\partial f} \gtreqless 0 \text{ as } \rho \gtreqless m^2 - 1, \text{ i.e. as } \epsilon \gtreqless \frac{m}{m+1}$$
$$\frac{\partial \epsilon}{\partial f} \gtreqless 0 \text{ as } \rho \gtreqless m^2 - 1, \text{ i.e. as } \epsilon \gtreqless \frac{m}{m+1}$$

If $\rho \geq 1$ then

$$\frac{\partial \rho}{\partial m} \gtreqless 0 \text{ as } \rho \gtreqless m^2 - 1, \text{ i.e. as } \epsilon \gtreqless \frac{m}{m+1}$$
$$\frac{\partial \epsilon}{\partial m} \gtreqless 0 \text{ as } \rho \gtreqless m^2 - 1, \text{ i.e. as } \epsilon \gtreqless \frac{m}{m+1}$$

If $f < 2.544948$ then

$$\frac{\partial \rho}{\partial m} < 0 \text{ if } m \leq 1 \text{ and } \rho < 1$$
$$\frac{\partial \epsilon}{\partial m} < 0 \text{ if } m \leq 1 \text{ and } \rho < 1$$

Proof. Let $q(k; f; m) = f - k^{\frac{1-m}{m}}\left[\frac{m+k+1}{m}\right]$. Then ρ is defined implicitly by $q \equiv 0$, provided that $q_k \neq 0$. Now $q_k = \frac{k^{\frac{1-2m}{m}}}{m}(m^2 - 1 - k)$. Consequently $q_k \lesseqgtr 0$ as $k \gtreqless m^2 - 1$ (since we assume throughout that $k > 0$). Furthermore

$q_f = 1$ and so $\frac{\partial \alpha}{\partial f} = -\frac{q_f}{q_k} \to 0$ as $k \to m^2 - 1$. We note from equation 40b that $\theta = \frac{1}{\frac{m}{\alpha+1}+1}$. It follws that $\alpha + 1 \to m^2$ if and only if $\theta \to \frac{m}{m+1}$. The derivatives for θ follow, since θ increases monotonically with α.

We have $q_m = \frac{k^{\frac{1-m}{m}}}{m^3}\left[m + mk + \ln k\,(k + m + 1)\right]$. Consequently

$$\frac{\partial \alpha}{\partial m} = \frac{k\,(m + mk + (k + m + 1)\ln k)}{m\,(k + 1 - m^2)} \tag{44}$$

. It is evident that if $k > 1$, then $\frac{\partial \alpha}{\partial m} \to 0$ as $k \to m^2 - 1$.

Consider the case $0 < m < 1$. On this interval we definitely have $k > m^2 - 1$. If $k < e^{-m}$ we see immediately that $\frac{\partial \alpha}{\partial m} < 0$. Note also that $k \to 1$ as $m \to 0$. It is clear therefore that on some interval to the right of $m = 0$ we will have $k < 1$ and $\frac{\partial \alpha}{\partial m} < 0$.

Now we can write $\frac{m+mk+(k+m+1)\ln k}{m}$ as $1 + k + f k^{\frac{m-1}{m}} \ln k$ (utilising the fact that we are differentiating along the curve defined by $q = 0$). If we let $z = 1 + k + f k^{\frac{m-1}{m}} \ln k$, then $\frac{\partial z}{\partial m} = \frac{\partial k}{\partial m} + k^{-\frac{1}{m}}\left[\frac{fk(\ln k)^2}{m^2} + f\left(\frac{m-1}{m}\right)\frac{\partial k}{\partial m}\ln(k) + f\frac{\partial k}{\partial m}\right]$. If $m < 1$ and $k < 1$, and $\frac{\partial k}{\partial m} > 0$, then every one of the terms in the square bracket is positive. It follows that if $\frac{\partial \alpha}{\partial m}$ is positive for any $m_0 < 1$, i.e. the numerator of the right hand side of equation 44 is positive then it will continue to be so for any $m \in (m_0; 1)$. Now when $m = 1$, it follows from equation 40a that $\alpha = f - 2$. This evidently has solutions with $k > 0$ only if $f > 2$. If we substitute in $\alpha = f - 2$ into equation 44 and evaluate this expression at $m = 1$, we find that $\frac{\partial \alpha}{\partial m} = f - 1 + f\ln(f - 2)$. $\frac{\partial \alpha}{\partial m} = 0$ if $f \simeq 2{:}544\,948\,12$. It follows that if $f < 2{:}544\,948\,12$, then $\frac{\partial \alpha}{\partial m} < 0$ on the interval $[0; 1]$. If $f > 2{:}544\,948\,12$ then there is a turning point for some $m \in (0; 1)$.

Note that for $f < 2$ we will have $k \to 0$ as $m \to 1$. This follows from the fact that we can write k as $k = \left(\frac{fm}{m+k+1}\right)^{\frac{m}{1-m}}$, i.e. $k < \left(\frac{fm}{m+1}\right)^{\frac{m}{1-m}}$. On this interval $\frac{fm}{m+1} < 1$ and hence $\lim_{m \to 1}\left(\frac{fm}{m+1}\right)^{\frac{m}{1-m}} = 0$.

The results for θ follow as before. ■

A.5 The multi-player game

As indicated in the text, we assume that the players are indexed from 1 to n. We assume that the aggregate production function is Y_A which is homogeneous of degree zero in $\alpha_1, \ldots, \alpha_n$. and that the appropriation functions can be written as

$$g_i = \frac{\gamma_i}{\sum_{j=1}^{n} \gamma_j}$$

where each γ_i is a function that depends on $(1 - \alpha_i)$ but not on any of the other α_j's and where g_i is assumed to be homogeneous of degree zero in the $(1 - \alpha_j)$ terms. Note that this is a generalisation of our assumption in the text that $\gamma_i = s_i^m$. We let

$$\gamma_{-i} = \sum_{j \neq i} \gamma_j$$

Lemma 36 $(1 - \alpha_i) \frac{\partial \gamma_i}{\partial \alpha_i} = -m\gamma_i$ for all $i = 1 \ldots n$

Proof. This proof follows the proof of Lemma 2 mutatis mutandis. ■

Proposition 37 $\frac{\partial Y_i}{\partial \alpha_i} = 0$ if and only if $\frac{\gamma_{-i}}{\gamma_i + \gamma_{-i}} Y_A = \frac{(1-\alpha_i)}{m} \frac{\partial Y_A}{\partial \alpha_i}$

Proof. By definition $Y_i = g_i Y_A$, i.e. $Y_i = \frac{\gamma_i}{\gamma_i + \gamma_{-i}} Y_A$. Differentiating this and noting that $\frac{\partial \gamma_{-i}}{\partial \alpha_i} = 0$, we get

$$\frac{\partial Y_i}{\partial \alpha_i} = \frac{\gamma_{-i} \frac{\partial \gamma_i}{\partial \alpha_i}}{\left(\gamma_i + \gamma_{-i}\right)^2} Y_A + \frac{\gamma_i}{\gamma_i + \gamma_{-i}} \frac{\partial Y_A}{\partial \alpha_i}$$

Substituting in the result of the previous lemma we get

$$\frac{\partial Y_i}{\partial \alpha_i} = \frac{\gamma_i}{\gamma_i + \gamma_{-i}} \left(-\frac{\gamma_{-i}}{\gamma_i + \gamma_{-i}} \frac{m Y_A}{(1 - \alpha_i)} + \frac{\partial Y_A}{\partial \alpha_i} \right)$$

The result follows given the restriction that $\gamma_i \neq 0$. ■

Proposition 38 At any interior Cournot equilibrium we have $Y_A = \frac{1}{mn - m + 1} \sum_{i=1}^{n} \frac{\partial Y_A}{\partial \alpha_i}$

Proof. At an interior Cournot equilibrium we must have $\frac{\partial Y_i}{\partial \alpha_i} = 0$ for $i = 1 \ldots n$, i.e.

$$\frac{\gamma_{-i}}{\gamma_i + \gamma_{-i}} Y_A = \frac{(1 - \alpha_i)}{m} \frac{\partial Y_A}{\partial \alpha_i} \text{ for every } i = 1 \ldots n$$

Consequently

$$\sum_{i=1} \frac{\gamma_{-i}}{\gamma_i + \gamma_{-i}} Y_A = \sum_{i=1} \frac{(1-\alpha_i)}{m} \frac{\partial Y_A}{\partial \alpha_i}$$

Now $\gamma_i + \gamma_{-i}$ is the same quantity for every i. The left hand side of the above expression therefore simplifies to

$$\frac{Y_A}{\gamma_i + \gamma_{-i}} \sum_{i=1} \gamma_{-i}$$

γ_{-i} contains every term except γ_i, so $\sum_{i=1}^{n} \gamma_{-i} = (n-1) \sum_{i=1}^{n} \gamma_i = (n-1)\left(\gamma_i + \gamma_{-i}\right)$. The left-hand side is therefore $(n-1)Y_A$. The right hand side is

$$\frac{1}{m} \sum_{i=1} \frac{\partial Y_A}{\partial \alpha_i} - \frac{1}{m} \sum_{i=1} \alpha_i \frac{\partial Y_A}{\partial \alpha_i}$$

By Euler's theorem we have $\sum_{i=1}^{n} \alpha_i \frac{\partial Y_A}{\partial \alpha_i} = Y_A$ and hence

$$(n-1)Y_A = \frac{1}{m} \sum_{i=1} \frac{\partial Y_A}{\partial \alpha_i} - \frac{Y_A}{m}$$

Simplifying, we get the result. ■

We apply some of these results to the specific context in which we have n identical players of type A and one player of type B (i.e. there are $n+1$ players in this game). Because the identical players all face identical choices, the optimal α_i's must all be the same. In a Cournot equilibrium, therefore these choices of α must satisfy any one of the typical reaction functions of a player A, i.e. we must have

$$\frac{(1-\beta)^m + (n-1)(1-\alpha)^m f^m k^m}{(1-\beta)^m + n(1-\alpha)^m f^m k^m} c(np\alpha k + \beta) W = \frac{(1-\alpha)}{m} cpkW \quad (45)$$

where we have used proposition 37. Player B's reaction function can be written as

$$\frac{n(1-\alpha)^m f^m k^m}{(1-\beta)^m + n(1-\alpha)^m f^m k^m} c(np\alpha k + \beta) W = \frac{1-\beta}{m} cW \quad (46)$$

Dividing equation 45 by 46 we get

$$\frac{(1-\beta)^m + (n-1)(1-\alpha)^m f^m k^m}{n(1-\alpha)^m f^m k^m} = \frac{(1-\alpha)}{(1-\beta)} pk$$

which we can write equivalently as

$$\frac{(1-\beta)^{m+1}}{(1-\alpha)^{m+1} f^{m+1} k^{m+1}} + (n-1)\frac{(1-\beta)}{(1-\alpha) f k} = n\frac{p}{f} \tag{47}$$

Now if we convert payoffs to wealth according to equation 6 then the new ratio of a player A's wealth to player B's new wealth will be given by

$$k_{t+1} = \frac{g_A}{g_B}$$

which will still be given by equation 7, i.e.

$$k_{t+1} = \frac{(1-\alpha)^m f^m k_t^m}{(1-\beta)^m}$$

The condition for intertemporal equilibrium, as before is that $k_{t+1} = k_t$. Using the fact that $k = \frac{(1-\alpha)^m f^m k^m}{(1-\beta)^m}$, we can rewrite equation 47 as

$$\left(\frac{1}{k}\right)^{\frac{m+1}{m}} + (n-1)\left(\frac{1}{k}\right)^{\frac{1}{m}} = n\frac{p}{f}$$

i.e.

$$\frac{1}{k} + (n-1) = n\frac{p}{f}k^{\frac{1}{m}} \tag{48}$$

This condition in fact defines k uniquely - the left hand side is decreasing in k and the right hand side increasing, so there can be at most one k satisfying this condition. Now let $Q(k) = \frac{1}{k} + (n-1) - n\frac{p}{f}k^{\frac{1}{m}}$. Then $Q = 0$ can be used to define k implicitly. Near $k = 0$ we have $Q(k) > 0$ while for k sufficiently large $Q(k) < 0$. We note that $Q\left(\frac{1}{n}\left(\frac{f}{p}\right)^{\frac{m}{m+1}}\right) = n - 1 + n\left(\frac{p}{f}\right)^{\frac{m}{m+1}}\left(1 - \frac{1}{n}^{\frac{1}{m}}\right) > 0$, i.e. $k > \frac{1}{n}\left(\frac{f}{p}\right)^{\frac{m}{m+1}}$. Furthermore $Q\left(\left(\frac{f}{p}\right)^{\frac{m}{m+1}}\right) = \left(\left(\frac{p}{f}\right)^{\frac{m}{m+1}} - 1\right)(1-n)$, so if $\frac{p}{f} > 1$ we have $Q < 0$ and hence $k < \left(\frac{f}{p}\right)^{\frac{m}{m+1}}$.

To get the comparative statics we note that $Q_k = -\frac{1}{k^2} - \frac{n}{m}\frac{p}{f}k^{\frac{1}{m}-1} < 0$, while $Q_n = 1 - \frac{p}{f}k^{\frac{1}{m}}$.

Now the condition $Q = 0$ can be written as $\left(\frac{1}{k} - 1\right) + n\left(1 - \frac{p}{f}k^{\frac{1}{m}}\right) = 0$. It follows that if $k < 1$ we must have $1 - \frac{p}{f}k^{\frac{1}{m}} < 0$, otherwise equality could not hold. Consequently $Q_n < 0$ whenever $k < 1$. In this case $\frac{\partial k}{\partial n} = -\frac{Q_n}{Q_k} < 0$.

To see what happens to output, we use Proposition 38. Given our choice of production functions we must have at every interior Cournot equilibrium

$$Y_A = \frac{c\,(npk + 1)\,W}{mn + 1}$$

(noting that the number of players is $n + 1$). Differentiating this expression with respect to n we get that

$$\frac{\partial Y_A}{\partial n} = \frac{c\,(pk - m)\,W}{(mn + 1)^2}$$

A.6 Property rights

We assume that the economy is divided into two sectors: an open sector, which is subject to predatory behaviour and a secure one which is not. The overall returns to each player are the sum of the two outputs, i.e.

$$R_i = Y_{s_i} + Y_{o_i}$$

We will assume that

$$Y_{s_i} = r_i \phi W_i$$

Once we have reparameterised, this implies that the returns are given by

$$\begin{aligned} R_A &= r_1 \phi k W + (1 - \phi)\,Y_1 \\ R_B &= r_2 \phi W + (1 - \phi)\,Y_2 \end{aligned}$$

where Y_1 and Y_2 are exactly the same as in equations 5a and 5b. With the choices of functional forms as given in equation 19, the payoffs can be written as

$$\begin{aligned} R_A &= \phi c^{\theta} p^{\theta} k W + (1 - \phi)\,Y_1 \\ R_B &= \phi c^{\theta} W + (1 - \phi)\,Y_2 \end{aligned}$$

Proposition 39 At an interior intertemporal Cournot equilibrium [illegible] we have

$$\hat{k} = \frac{(1-\phi)cpf^{\frac{m}{m+1}} + \left(p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}\right)(m+1)(r_1-r_2)\phi - (1-\phi)cp^{\frac{m}{m+1}}}{2(1-\phi)cp^{\frac{2m+1}{m+1}}} + \frac{\sqrt{\left((1-\phi)cpf^{\frac{m}{m+1}} + \left(p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}\right)(m+1)(r_1-r_2)\phi - (1-\phi)cp^{\frac{m}{m+1}}\right)^2 + 4(1-\phi)^2c^2p^{\frac{2m+1}{m+1}}f^{\frac{m}{m+1}}}}{2(1-\phi)cp^{\frac{2m+1}{m+1}}} \quad (49a)$$

$$\hat{\alpha} = 1 - \frac{m}{m+1}\left(\frac{p^{\frac{m}{m+1}}}{p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}}\right)\frac{p\hat{k}+1}{p\hat{k}} \quad (49b)$$

$$\hat{\beta} = 1 - \frac{m}{m+1}\left(\frac{f^{\frac{m}{m+1}}}{p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}}\right)\left(p\hat{k}+1\right) \quad (49c)$$

At a corner intertemporal equilibrium $\left(\hat{\alpha}; 0; \hat{k}\right)$ the equilibrium α and k will satisfy the conditions

$$(1-\phi)c\frac{p}{f}[\alpha(m+1)-1]^{\frac{1}{m}} = m(r_1-r_2)\phi(1-\alpha)^{\frac{1}{m}} + (1-\phi)cp[\alpha(m+1)-1](1-\alpha)^{\frac{1}{m}} \quad (50a)$$

$$k = \frac{[\alpha(m+1)-1]^{\frac{1}{m}}}{(1-\alpha)^{\frac{m+1}{m}}f} \quad (50b)$$

Furthermore we require

$$k \geq \frac{1+(m+1)\left(\frac{p}{f}\right)^{\frac{m}{m+1}}}{mp}$$

At a corner intertemporal equilibrium $(0; \beta; k)$ the equilibrium β and k will satisfy the conditions

$$(1-\phi)c[\beta(m+1)-1](1-\beta)^{\frac{1}{m}} = m(r_1-r_2)\phi(1-\beta)^{\frac{1}{m}} + (1-\phi)c(\beta(m+1)-1)^{\frac{1}{m}}f \quad (51)$$

$$k = \frac{(1-\beta)^{\frac{m+1}{m}}}{(\beta(m+1)-1)^{\frac{1}{m}}f} \quad (52)$$

Furthermore we require

$$k \le \frac{m}{p\left(1+(m+1)\left(\frac{f}{p}\right)^{\frac{m}{m+1}}\right)}$$

Proof. By equation 18 we have $k_{t+1} = \frac{R_{A;t}}{R_{B;t}}$. Substituting in for R_A and R_B from equations 17a and 17b and imposing the condition that $k_{t+1} = k_t$ we get

$$k = \frac{r_1\phi kW + (1-\phi)Y_1}{r_2\phi W + (1-\phi)Y_2}$$

Rearranging we get

$$(r_1 - r_2)\phi kW = (1-\phi)Y_2\left(k - \frac{Y_1}{Y_2}\right) \tag{53}$$

At any interior Cournot equilibrium, we have by Remark 23 that

$$\frac{Y_1}{Y_2} = \left(\frac{f}{p}\right)^{\frac{m}{m+1}}$$

and

$$Y_2 = \frac{p^{\frac{m}{m+1}}}{p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}}\frac{c(pk+1)W}{m+1}$$

Substituting these into equation 53, we get

$$(r_1 - r_2)\phi k = (1-\phi)\frac{p^{\frac{m}{m+1}}}{p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}}\frac{c(pk+1)}{m+1}\left[k - \left(\frac{f}{p}\right)^{\frac{m}{m+1}}\right]$$

i.e.

$$(r_1 - r_2)\phi k = (1-\phi)\frac{c(pk+1)}{\left(p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}\right)(m+1)}\left[kp^{\frac{m}{m+1}} - f^{\frac{m}{m+1}}\right]$$

Multiplying out and rearranging we get a quadratic in k:

$$(1-\phi)cp^{\frac{2m+1}{m+1}}k^2 - \left\{\left(p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}\right)(m+1)(r_1 - r_2)\phi - (1-\phi)cp^{\frac{m}{m+1}} + (1-\phi)cpf^{\frac{m}{m+1}}\right\}k - (1-\phi)cf^{\frac{m}{m+1}} = 0$$

This quadratic will always have real roots. Let

$$
\begin{aligned}
a &= (1-\phi)\,cp^{\frac{2m+1}{m+1}} \\
b &= \left\{ (1-\phi)\,cpf^{\frac{m}{m+1}} + \left(p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}\right)(m+1)(r_1-r_2)\phi - (1-\phi)\,cp^{\frac{m}{m+1}} \right\} \\
c &= (1-\phi)\,cf^{\frac{m}{m+1}}
\end{aligned}
$$

Then our solution will be given by

$$k = \frac{b + \sqrt{b^2 + 4ac}}{2a}$$

since the other root will be negative ($4ac > 0$), i.e.

$$k = \frac{(1-\phi)\,cpf^{\frac{m}{m+1}} + \left(p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}\right)(m+1)(r_1-r_2)\phi - (1-\phi)\,cp^{\frac{m}{m+1}}}{2(1-\phi)\,cp^{\frac{2m+1}{m+1}}} + \frac{\sqrt{\left\{(1-\phi)\,cpf^{\frac{m}{m+1}} + \left(p^{\frac{m}{m+1}} + f^{\frac{m}{m+1}}\right)(m+1)(r_1-r_2)\phi - (1-\phi)\,cp^{\frac{m}{m+1}}\right\}^2 + 4(1-\phi)^2 c^2 p^{\frac{2m+1}{m+1}} f^{\frac{m}{m+1}}}}{2(1-\phi)\,cp^{\frac{2m+1}{m+1}}}$$

The results for α and β merely repeat the results in equations 32a and 32b.

Consider now a corner solution $(\alpha; 0; k)$. From the results in remark 25 we get that $\frac{Y_2}{W} = \frac{c(1-\alpha)pk}{m}$ and that $\frac{Y_1}{Y_2} = \frac{\alpha(m+1)-1}{1-\alpha}$. The condition in equation 53 must also hold for this case. Substituting in we get

$$(r_1-r_2)\phi k = (1-\phi)\,\frac{c(1-\alpha)pk}{m}\left(k - \frac{\alpha(m+1)-1}{1-\alpha}\right)$$

Dividing out k and simplifying we get

$$(1-\phi)\,c(1-\alpha)pk = m(r_1-r_2)\phi + (1-\phi)\,cp[\alpha(m+1)-1]$$

Now by Theorem 24 the corner solution will satisfy $(1-\alpha)^{m+1} f^m k^m = \alpha(m+1)-1$. We can solve for k in this expression. In particular we note that $(1-\alpha)k = \left[\frac{\alpha(m+1)-1}{1-\alpha}\right]^{\frac{1}{m}}\frac{1}{f}$. Substituting this into the expression above and simplifying we get

$$(1-\phi)c\frac{p}{f}[\alpha(m+1)-1]^{\frac{1}{m}} = m(r_1-r_2)\phi(1-\alpha)^{\frac{1}{m}} + (1-\phi)cp[\alpha(m+1)-1](1-\alpha)^{\frac{1}{m}}$$

The requirement that $k \geq \frac{1+(m+1)\left(\frac{p}{f}\right)^{\frac{m}{m+1}}}{mp}$ follows also from Theorem 24.

In the case of a corner $(0;\beta;k)$ we also use remark 25. We have that $\frac{Y_2}{W} = \frac{c[\beta(m+1)-1]}{m}$ and that $\frac{Y_1}{Y_2} = \frac{1-\beta}{\beta(m+1)-1}$. Substituting these expressions into equation 53 we get

$$(r_1-r_2)\phi k = (1-\phi)c\frac{\beta(m+1)-1}{m}\left(k - \frac{1-\beta}{\beta(m+1)-1}\right)$$

i.e.

$$(1-\phi)c[\beta(m+1)-1]k = m(r_1-r_2)\phi k + (1-\phi)c(1-\beta)$$

But at this corner β and k have to satisfy the condition $(1-\beta)^{m+1} = f^m k^m(\beta(m+1)-1)$, i.e.

$$k = \frac{(1-\beta)^{\frac{m+1}{m}}}{(\beta(m+1)-1)^{\frac{1}{m}}f}$$

Substituting this in and simplifying we get

$$(1-\phi)c[\beta(m+1)-1](1-\beta)^{\frac{1}{m}} = m(r_1-r_2)\phi(1-\beta)^{\frac{1}{m}} + (1-\phi)c(\beta(m+1)-1)^{\frac{1}{m}}f$$

The condition on k follows, as before, from Theorem 24. ■

Proposition 40 If, $r_1 \geq r_2$ then at any interior equilibrium and at any stable corner equilibrium an increase in ϕ from 0 will change the intertemporal k such that $k \geq k^*$ where k^* is the corresponding intertemporal equilibrium when $\phi = 0$. The inequalities will be reversed at any unstable corner equilibrium. The size of the shifts will increase with (r_1-r_2) and ϕ.

Proof. Equation 53 gives the condition

$$(r_1 - r_2)\phi kW = (1-\phi)Y_2\left(k - \frac{Y_1}{Y_2}\right)$$

If $\phi = 0$ then the intertemporal equilibrium is given by $k^* = \frac{Y_1}{Y_2}$. Now in an interior equilibrium $\frac{Y_1}{Y_2}$ is independent of k (by Remark 23) The result follows.for this case.

At the corner the situation is a bit more complex since $\frac{Y_1}{Y_2}$ will be a function of k also. Let $D(k) = k - \frac{Y_1(k)}{Y_2(k)}$. The solutions to $D(k) = 0$ will be all the intertemporal equilibria for the case $\phi = 0$. As we have noted in the proof of proposition 33 $D'(k^*) > 0$ if k^* is a stable equilibrium. If $m < 1$ then this will be the only equilibrium, i.e. $D(k)$ will be positive for all $k > k^*$ and negative for all $k < k^*$ the only solution to the condition above will therefore be for a $k > k^*$.

By differentiating again it is easy to show that $D''(k) = \frac{m}{(m+1)^2}\frac{\alpha(m+1)-1}{(1-\alpha)\alpha^3}\frac{2\alpha-1}{k^2}$. This will be positive if $\alpha > \frac{1}{2}$, which it will be at a stable equilibrium if $m > 1$. This proves that $D(k)$ will be positive for all $k > k^*$.

Now note that we can rewrite the above condition as

$$m(r_1 - r_2)\phi = (1-\phi)cp\left\{\left(\frac{\alpha(m+1)-1}{1-\alpha}\right)^{\frac{1}{m}}\frac{1}{f} - \alpha(m+1)+1\right\} \quad (54)$$

(compare this with equation 50a). At $(\alpha^*; 0; k^*)$ the term in braces is zero. By choosing an α sufficiently close to one, we can make the term in braces as large as we like. We can certainly make it larger than the left-hand side, for any given r_1, r_2, and ϕ. This implies that we can always find a $k > k^*$ such that $(r_1 - r_2)\phi kW < (1-\phi)Y_2\left(k - \frac{Y_1}{Y_2}\right)$. It follows (by the continuity of the functions) that there will be a solution with $k > k^*$ at which we will have $(r_1 - r_2)\phi kW = (1-\phi)Y_2\left(k - \frac{Y_1}{Y_2}\right)$.

It is clear that the larger the term $(r_1 - r_2)\phi$ is, the larger $D(k)$ has to be to ensure equality in equation 53.

The results for β follow by symmetry. ■

Proposition 41 Let $r_i = c_i^{\mu}$. If $\mu > 1$, then the intertemporal corner equilibrium $(\hat{\alpha}; 0; \hat{k})$ will behave as follows: as $p \to \infty$, $\hat{\alpha} \to 1$ and $\hat{k} \to \infty$. Furthermore as $c \to \infty$, $\hat{\alpha} \to 1$ and $\hat{k} \to \infty$.

Furthermore I_T and I_C (where these are as defined in the text) will tend to zero, as $p \to \infty$.

Proof. In this case we can write equation 54 as

$$mc^{\theta-1}\left(\frac{p^{\theta}-1}{p}\right)\phi = (1-\phi)\left\{\left(\frac{\alpha(m+1)-1}{1-\alpha}\right)^{\frac{1}{m}}\frac{1}{f}-\alpha(m+1)+1\right\}$$

As $p \to \infty$ the left hand side of this equation tends to infinity. Equality can hold only if $\alpha \to 1$. It follows from the equation for $\hat{k}$ that $\hat{k} \to \infty$. The result on c follows likewise.

To show what happens to I_T, we note that at any corner equilibrium, $I_T = g_B$ (this follows from the fact that $Y_B = 0$ and hence $\frac{Y_A - Y_1}{Y_A + Y_B} = 1 - \frac{Y_1}{Y_A} = 1 - \frac{g_A Y_A}{Y_A}$). From Remark 25 it follows that $g_B = \frac{(1-\alpha)}{\alpha m}$. Hence $I_T \to 0$ as $\alpha \to 1$.

By definition $I_C = (1-\alpha)\frac{k}{k+1} + (1-\beta)\frac{1}{k+1}$. As $k \to \infty$ and $\alpha \to 1$ it follows that $I_C \to 0$. ■