Random Expected Utility Theory with a Continuum of Prizes

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Abstract

This note generalizes Gul and Pesendorfer's random expected utility theory, a stochastic reformulation of von Neumann-Morgenstern expected utility theory for lotteries over a finite set of prizes, to the circumstances with a continuum of prizes. Let $[0, M]$ denote this continuum of prizes; assume that each utility function is continuous, let $C_0[0, M]$ be the set of all utility functions which vanish at the origin, and define a random utility function to be a finitely additive probability measure on $C_0[0, M]$ (associated with an appropriate algebra). It is shown here that a random choice rule is mixture continuous, monotone, linear, and extreme if, and only if, the random choice rule maximizes some regular random utility function. To obtain countable additivity of the random utility function, we further restrict our consideration to those utility functions that are continuously differentiable on $[0, M]$ and vanish at zero. With this restriction, it is shown that a random choice rule is continuous, monotone, linear, and extreme if, and only if, it maximizes some regular, countably additive random utility function. This generalization enables us to make a discussion of risk aversion in the framework of random expected utility theory.

1 Introduction

A multitude of experimental studies have shown that the behavior of an economic agent is not consistent with a deterministic utility function, but rather exhibits a certain kind of randomness. For example, it has been shown in May (1954) that when a subject is asked to reveal his preference between two objects, say x and y, he may prefer x to y in one occasion, and reverse this preference by preferring y to x in another. Moreover, Fishburn (1991) shows that an economic agent's choices may reveal preference cycles. Hence, x may be chosen over y in one occasion, y over z on another occasion and z might be chosen of x on yet another occasion. Yet another problem for deterministic choice behavior is demand theory: aggregate demand data necessitates a model of random choice. It is obvious that these phenomena are at variation with the fundamental tenet of deterministic utility theory that an individual preference be a weak order (or at least a preorder).

One may explain the randomness of individual behavior in several different ways. Among them are variations in tastes, incomplete and varied information on the alternatives, and errors of optimization by the agent (see for example McFadden (1980)). The first, i.e. variations in tastes, is

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taken by random utility theory. To be more specific let D be a finite set of alternatives and $\rho^D(x)$ the proportion of a group of individuals who choose x from D. To simplify the exposition we assume that the subjects involved be observationally indistinguishable, so that $\rho^D(x)$ represents the probability of a subject choosing x from D. We shall henceforth call ρ a random choice rule. Assume furthermore that the subject be a utility maximizer, so from his behavior of choosing x_1 from D, we can deduce his utility function (or taste) u satisfies $u(x_1) \geq u(z)$ for all $z \in D$. On the other hand, when he chooses a distinct element x_2 from D, we can explain his behavior by postulating that he has changed his utility function from u to v which satisfies $v(x_2) \geq v(z)$ for all $z \in D$.

More formally, let $\mathbf{U}$ be a certain set of relevant utility functions and $\mathfrak{U}$ an appropriate algebra of subsets of $\mathbf{U}$; then the fundamental question of random utility theory asks whether there exists on $(\mathbf{U}, \mathfrak{U})$ a finitely additive probability measure μ, which is customarily called a random utility function, such that $\rho^D(x)$ is equal to the μ-probability of choosing from $\mathbf{U}$ a u that attains its maximum at x in D for every D. If such a measure exists, we say that the random choice rule ρ maximizes the random utility function μ.

This question has a long history and its early attempts date back to, for instance, Block and Marschak (1960), Luce (1958), Thurstone (1927). More recently, further attempts have been made; among them there are, most notably, Clark (1996), Falmagne (1978), McFadden (2005). These three theories, when adapted to the domain of decision-making under risk and assuming each individual conform with von Neumann-Morgenstern (vNM) expected utility theory, have been shown in Gul and Pesendorfer (2006b) to be equivalent to the random expected utility theory of Gul and Pesendorfer (2006a), which, for easier reference, we shall call Gul-Pesendorfer theory.

More specifically, the Gul-Pesendorfer theory deals with decision-making under risk in the situation with a finite set of prizes; that is, the objects of choice are limited to lotteries over a finite set of prizes. This however excludes some interesting cases because in most economic models the prize space for lotteries with monetary prizes and for lotteries over interest rates is not finite. To cover such cases we shall in this paper take $\mathbf{I} = [0, M]$ as the set of possible prizes, and define the lottery as a Borel probability measures on it. The object of the paper is to generalize Gul-Pesendorfer theory to this situation. Recall that Gul-Pesendorfer theory states that a random choice rule maximizes some regular random utility function if, and only if, the random choice rule is monotone, mixture continuous, linear, and extreme; and that by strengthening mixture continuity to continuity, the resulting random utility function will be countably additive. We shall show in this paper that these statements are still true even in the case of a continuum of prizes describe above. As a side benefit, this generalization makes it possible for us to discuss the notion of risk aversion.

The outline of this paper is as follows. In Section 2, we introduce some notations and state formally the problem under consideration. Section 3 describes the conditions that a RCR should fulfill in order for it to maximize some regular random utility function, and Section 4 furthers this investigation by studying under what condition a RCR will maximize a regular, countably additive random utility function. In Section 5 we define and characterize the notion of risk aversion, and in Section 6 we make the proofs of all the results stated in the previous sections.

2 Statement of the Problem

As said in the Introduction, we let $\mathbf{I} = [0, M]$ denote a set of monetary prizes and $\mathfrak{B}(\mathbf{I})$ the Borel σ-algebra of subsets of $\mathbf{I}$. Let $\mathbf{X}$ be the space of all probability measures (which will henceforth be called lotteries) on $(\mathbf{I}, \mathfrak{B}(\mathbf{I}))$; we denote a generic lottery as x, y, etc., and associate with $\mathbf{X}$ the finite-cofinite algebra $\mathfrak{X}$, i.e.

$$\mathfrak{X} = \{B \subseteq \mathbf{X} | \text{ either } B \text{ is finite or } B^c \text{ is finite}\},$$

where B^c denotes the complement of B in $\mathbf{X}$. Let Π be the set of all finitely additive probability measures on $(\mathbf{X}, \mathfrak{X})$. A decision problem is defined to be a finite subset of $\mathbf{X}$; let $\mathfrak{D}$ be the set of all decision problems, and we shall denote its generic element as D, D', etc. A random choice rule (RCR), denoted by ρ, is a mapping from $\mathfrak{D}$ to Π; it specifies for each decision problem the probability of choosing a lottery in B for each $B \in \mathfrak{X}$. For example, given a decision problem D, $\rho^D(\{x\})$ denotes the probability of choosing x; for the simplicity of notation we shall write $\rho^D(x)$ for $\rho^D(\{x\})$. It is natural to require $\rho^D(x) = 0$ for every $x \notin D$; in other words, we shall require throughout this paper that $\rho^D(D) = 1$ for every $D \in \mathfrak{D}$.

Let $\mathbf{U}$ be a set of continuous utility functions on $\mathbf{I}$. We assume that the economic agent behaves in line with vNM expected utility theory, i.e., for each $x \in \mathbf{X}$, the utility level it affords is measured by

$$u_e(x) = \int_0^M u(t)dx(t), \text{where } u \in \mathbf{U}. \tag{2.1}$$

This means that if the agent chooses x from D, then his utility function must be a member of the set

$$N(D, x) = \{u \in \mathbf{U} | \ u_e(x) \geq u_e(z), \forall z \in D\}.$$

Let $\mathcal{K}^*$ be the set of all $N(D, x)$'s, i.e. $\mathcal{K}^* = \{N(D, x) | \ x \in D, D \in \mathfrak{D}\}$, and let $\mathfrak{U}$ be the algebra generated by $\mathcal{K}^*$. Then, a random utility function (RUF) is a finitely additive probability measure on $(\mathbf{U}, \mathfrak{U})$.

In the following we shall restrict our consideration to a special kind of RUF's: regular RUF's. Let

$$N^+(D, x) = \{u \in \mathbf{U} | \ u_e(x) > u_e(z), \forall z \in D \backslash \{x\}\};$$

that is, $N^+(D, x)$ is the set of utility functions that have x as the unique maximizer in D. It is easily seen that $N^+(D, x) \cap N^+(D, y) = \varnothing$ for any distinct x, y in D. Then a RUF μ is regular if

$$\mu(\cup_{x \in D} N^+(D, x)) = 1.$$

In words, a RUF is regular if the realized utility function has a unique maximizer with μ-probability one. When $\mathbf{U} = C(\mathbf{I})$, the existence of a regular RUF on $(\mathbf{U}, \mathfrak{U})$ is given in the Appendix. Our basic object of study is then given by the following

Definition 2.1 A RCR ρ maximizes a regular RUF μ if $\rho^D(x) = \mu(N(D, x))$ for all $D \in \mathfrak{D}$ and $x \in \mathbf{X}$.

It is useful to note, and this is easily verified because D is finite, that this definition can be simplified to $\rho^D(x) = \mu(N(D, x))$ for all $D \in \mathfrak{D}$ and $x \in D$. Concerning the relation between a

RCR and a RUF, we have first the following result (cf. Gul and Pesendorfer (2006a, Theorem 1, p. 126)):

THEOREM 2.1 Every regular RUF has a unique RCR as its maximizer, and every RCR can maximize at most one regular RUF.

It should be pointed out that this theorem holds valid only if **U** consists exclusively of continuous functions. In other words, if **U** is allowed to include discontinuous functions, the theorem would no longer be valid; for an example see Appendix A.2.[1]

It is not hard to observe that this theorem does not answer the question whether there exists for every RCR a regular RUF which has the RCR as its maximizer. This question is the theme of the next two sections: Section 3 describes the conditions that a RCR should satisfy in order for it to maximize some RUF, and Section 4 sharpens the conditions of Section 3 to ensure that the maximized RUF is countably additive.

3 Identification of Random Utility Functions

This section describes the properties that a RCR should satisfy in order to maximize some RUF. We shall take **U** to be the set of continuous functions on **I** which are normalized to be zero at the origin, that is,

$$\mathbf{U} = \{u \in C(\mathbf{I}) | u(0) = 0\},$$

where, recall, $C(\mathbf{I})$ denotes the set of continuous functions on **I**. We know from functional analysis that not every pair of lotteries can be separated by an element of **U** or, in other words, there exist two lotteries which yield the same level of expected utility for all elements of **U**. Such lotteries can be regarded as one and the same, because the only concern of the economic agent is with the expected utility of a lottery. To describe this formally we shall in Section 6 introduce the notion of quotient space, of which every element is a set of lotteries that yield the same level of expected utility for all elements of **U** and every two distinct elements yield different levels of expected utility for at least one element of **U**.

There are four properties to be described, among which the first is concerned with continuity. To define it we have to introduce topological structures on $\mathfrak{D}$ and Π. Observing that Π is simply a set of probability measures, we can (and shall) associate with it the topology of weak convergence. For $\mathfrak{D}$ we impose on it the Hausdorff topology which is generated by the Hausdorff metric

$$d_h(D, D') = \max\{\max_{x\in D} \min_{x'\in D'} \|x - x'\|_v, \max_{x'\in D'} \min_{x\in D} \|x - x'\|_v\} \text{ for any } D, D' \in \mathfrak{D},$$

where $\|\cdot\|_v$ stands for the total variation norm.

With this topological structure on $\mathfrak{D}$ we can now define the notion of continuity. Recall that for D, D' in $\mathfrak{D}$ and α in $[0, 1]$, $\alpha D + (1-\alpha)D' = \{\alpha x + (1-\alpha)x' |\ x \in D, x' \in D'\}$.

DEFINITION 3.1 The RCR ρ is said to be mixture continuous if $\rho(\alpha D + (1-\alpha)D')$ is continuous in α for all D, D' in $\mathfrak{D}$.

[1] Thanks for one of the reviewers for pointing this out and providing the example.

The understanding of this notion of continuity may be facilitated by analogy with the weak continuity of an individual preference preorder $\succeq$. Recall that $\succeq$ is said to be weakly continuous if $\{\alpha \in [0,1] |\ \alpha x_1 + (1-\alpha)x_2 \succeq \alpha z_1 + (1-\alpha)z_2\}$ is closed in [0, 1] for all x_i, z_i, $i = 1, 2$. It is then clear that mixture continuity of a RCR can be thought of as a stochastic analogue of the weak continuity of $\succeq$.

The second property is that of monotonicity. This property is rather intuitively appealing; it states that the probability of choosing a certain alternative from a decision problem D should not be increased with the number of alternatives in D. More formally,

Definition 3.2 The RCR ρ is monotone if $\rho^{D'}(x) \leq \rho^D(x)$ for $x \in D \subseteq D'$.

The third one is a stochastic analogue of the independence axiom of vNM expected utility theory. Recall that this axiom states that if an individual prefers x to z, then he should also prefers $\alpha x + (1-\alpha)y$ to $\alpha z + (1-\alpha)y$ for $\alpha \in (0,1)$ and $y \in \mathbf{X}$. In probabilistic terms, this means that the probability of an individual choosing x in D should be the same as the probability of his choosing $\alpha x + (1-\alpha)y$ in $\alpha D + (1-\alpha)\{y\}$. More formally,

Definition 3.3 The RCR ρ is linear if for all $x \in D$, $y \in \mathbf{X}$, and $\alpha \in (0,1)$

$$\rho^{\alpha D + (1-\alpha)\{y\}}(\alpha x + (1-\alpha)y) = \rho^D(x).$$

The last one states that only extreme points have a chance to be selected. More formally, let ext$D$ be the set of extreme points of the convex hull of D for any $D \in \mathfrak{D}$; then

Definition 3.4 The RCR ρ is extreme if $\rho^D(\text{ext}D) = 1$.

Now we can state the main result of this section:

Theorem 3.1 A RCR is mixture continuous, monotone, linear, and extreme if and only if the RCR maximizes some regular RUF.

4 Countably Additive Random Utility Function

Recall that a RUF is, by definition, finitely additive. It is desirable, and sometimes even necessary, to have countable additivity. So this section aims to find out the conditions on a RCR in order for it to be able to identify a countably additive RUF.

In the case of finite prizes, the work of Gul and Pesendorfer (2006a) reveals that this can be achieved by strengthening mixture continuity to continuity. This achievement is based (among others) on the fact that the unit sphere of a finite-dimensional Euclidean space is compact. It is well known however that the unit sphere of an infinite-dimensional metric space is not compact. For this reason we shall in this section restrict our consideration to the space of continuously differentiable functions u on $\mathbf{I}$ with $u(0) = 0$. Formally we shall take

$$\mathbf{U} = \{u \in C^1(\mathbf{I}) |\ u(0) = 0\}, \tag{4.1}$$

where, recall, $C^1(\mathbf{I})$ denotes the space of continuously differentiable functions on $\mathbf{I}$. The restriction to differentiable utility functions is not only for technical reasons: it has economic content. For instance, Nakamura (2015) established that if a risk-averse economic agent exhibits a preference for small positive risk taking, then his utility function must be differentiable. Here a small positive risk is a lottery whose expectation is positive and whose support is contained in an interval $(-\epsilon, \epsilon)$ with ϵ being sufficiently small.

This restriction together with Gul and Pesendorfer's notion of continuity mentioned above allows us to identify a countably additive RUF. We first recount the formal definition of continuity:

DEFINITION 4.1 A RCR ρ is said to be continuous if ρ^D is continuous in D for all $D \in \mathfrak{D}$.

Just as mixture continuity, the understanding of continuity may be facilitated by analogy with strong continuity of an individual preference preorder $\succeq$. Recall that $\succeq$ is said to be strongly continuous if $x_n \to x$, $y_n \to y$, and $x_n \succeq y_n$, then $x \succeq y$ (see for instance Dubra et al. (2004)). It is clear that continuity of a RCR can be thought of as a stochastic analogue of the strong continuity of $\succeq$.

Armed with this concept we can now state

THEOREM 4.1 A RCR is continuous, monotone, linear, and extreme if and only if the RCR maximizes some regular, countably additive RUF.

We remark that following the proof of Theorem 2.1, one can demonstrate that the regular, countably additive RUF that is maximized by a RCR is unique.

5 Risk Aversion

This section studies the notion of risk aversion in the context of random expected utility theory. We first present two definitions, which are apparently equally intuitive and reasonable, of comparative risk aversion: one in terms of RCR and the other in terms of RUF, and then examine their relationship. We conclude the section with a discussion of Hilton (1989)'s work.

For convenience of notation we denote the risk-free lottery δ_t (the Dirac measure at $t \in \mathbf{I}$) by t. Let $\mathfrak{D}_t$ be the set of decision problems which contain t as an alternative. Assume given two individuals, 1 and 2, and that individual i is associated with a RCR ρ_i which maximizes a regular, countably additive RUF μ_i, $i = 1, 2$. With random utility theory, it appears reasonable to say that individual 1 is more risk averse than individual 2 at $t \in (0, M)$ if individual 1 always chooses t from D with a higher probability than individual 2 for any $D \in \mathfrak{D}_t$. Here we consider an open interval $(0, M)$ because it seems harmless to assume that any lottery is preferred to 0 and less preferred to M. Formally,

DEFINITION 5.1 Individual 1 is said to be more risk averse than individual 2 at $t \in (0, M)$ if $\rho_1^D(t) \geq \rho_2^D(t)$ for any $D \in \mathfrak{D}_t$.

We proceed to define comparative risk aversion in terms of RUF. Recall that, in the case of deterministic utility, individual 1 is more risk averse than individual 2 if, and only if, individual 1's utility function has a larger Arrow-Pratt measure than that of individual 2. Therefore, with random utility, it seems equally reasonable to say that individual 1 is more risk averse than individual 2 if it is more probable for individual 1 than for individual 2 to realize (or invoke) a utility function with a larger Arrow-Pratt measure.

To make precise the term “more probable,”let

$$r(u,t) = -\frac{u''(t)}{u'(t)}$$

be the Arrow-Pratt measure of $u \in \mathbf{U}$ at t. It induces a probability measure on the real line: for any real number r_0,

$$F_i^t(r_0) = \mu_i(A^t(r_0)), i = 1, 2, \tag{5.1}$$

where we have put $A^t(r_0) = \{u \in \mathbf{U} | r(u,t) \leq r_0\}$. Assume for the moment that $A^t(r_0)$ is measurable, so that F_i^t is well define. Then the term “more probable”can be formalized as F_1^t first-order stochastically dominating F_2^t. Formally

DEFINITION 5.2 Individual 1 is said to have a greater stochastic coefficient of absolute risk aversion than individual 2 at $t \in (0, M)$ if F_1^t first-order stochastically dominates F_2^t.

Although both definitions appear reasonable, it is not easy to establish their equivalence in general. For this reason as well as for the reason of establishing the measurability of $A^t(r_0)$ we shall restrict our attention to a special set of utility functions, namely, a set $\mathbf{U} \subset C^2(\mathbf{I})$ of concave functions in which any function is an increasing strictly concave transformation of another function in it, where, recall, $C^2(\mathbf{I})$ denotes the set of twice continuously differentiable functions on $\mathbf{I}$. Formally we define $\mathbf{U} \subset C^2(\mathbf{I})$ to be the set of concave functions such that

(i) If $u \in \mathbf{U}$, then any increasing strictly concave transformation of u belongs to $\mathbf{U}$;
(ii) If $u, v \in \mathbf{U}$, then u is an increasing strictly concave transformation of v, or vice versa.

In particular, $\mathbf{U}$ may be so constructed as to contain the set of CARA functions. With this restriction we claim that

LEMMA 5.1 The set $A^t(r_0)$ is measurable for all nonnegative r_0 and all $t \in (0, M)$.

We can now state the main result of this section:

THEOREM 5.1 The following are equivalent:

(i) Individual 1 is more risk averse than individual 2 at $t \in (0, M)$;
(ii) Individual 1 has a greater stochastic coefficient of absolute risk aversion than individual 2 at $t \in (0, M)$.

To conclude this section we make a brief discussion of Hilton (1989)’s work. Hilton studied risk aversion in the framework of random expected utility theory with a discrete[2] set of utility functions, and presented five alternative definitions of comparative risk aversion: one in terms of RCR and the other four in terms of RUF. He then examined the relationship between the four definitions in terms of RUF, but without investigating their relationship with the one in terms of RCR. In this connection, the present section could be viewed as a complement to Hilton (1989)’s work.

[2]The term “discrete”is not explicitly defined in Hilton (1989); I understand it to mean either finite or countable.

5.1 Proof of Lemma 5.1 and Theorem 5.1

Let us begin by proving Lemma 5.1. For this fix t, r_0, and let

$$A_1 = \{u \in \mathbf{U} | r(u,t) < r_0\},\ A_2 = \{u \in \mathbf{U} | r(u,t) = r_0\}, \tag{5.2}$$

so that $A^t(r_0) = A_1 \cup A_2$. It suffices to show the measurability of A_1 and A_2.

Let us begin with A_2. Since in $\mathbf{U}$, any function is an increasing strictly concave transformation of another, it follows that A_2 is a singleton, say $A_2 = \{u_0\}$. Take $x_0 \in \mathbf{X}$ and consider the hyperplane

$$H = \{x \in \mathbf{X} | \langle u_0, x \rangle = \langle u_0, x_0 \rangle\}$$

Since $\mathbf{X}$ associated with the Prokhorov metric is separable, so is H; let $\{x_1, x_2, \ldots\}$ be a countable dense subset of H. Define

$$D_n = \{x_{2n-1}, x_{2n}, y_n\}, n = 1, 2, \ldots,$$

where $y_n = \frac{1}{2}x_{2n-1} + \frac{1}{2}x_{2n}$. It is not hard to see that $u_0 = \cap_{n=1}^{\infty} N(D_n, y_n)$, hence that A_2 is measurable.

For the measurability of A_1, we note that $r_0 = 0$ implies $A_1 = \varnothing$, which is trivially measurable. Now suppose that $r_0 > 0$ and let

$$B_n = \{u \in \mathbf{U} | r(u,t) \le r_0 - \frac{1}{n}\}, n = 1, 2, \ldots, \tag{5.3}$$

so that $B_1 \subset B_2 \subset B_3 \subset \cdots$ and $A_1 = \cup_{n=1}^{\infty} B_n$. It suffices therefore to show the measurability of each B_n. Again if $B_n = \varnothing$, it is trivially measurable. So assume $B_n \neq \varnothing$ and therefore there exists an $u^n \in \mathbf{U}$ such that $r(u^n, t) = r_0 - 1/n$. Take a nondegenerate $z_n \in \mathbf{X}$ such that $u_e^n(t) = u_e^n(z_n)$, where u_e^n is as defined in (2.1). Then applying Jensen's inequality we can conclude that $B_n = N(E_n, z_n)$, where $E_n = \{z_n, t\}$, and therefore B_n is measurable. This concludes the proof of Lemma 5.1.

We turn now to the proof of Theorem 5.1. We begin by proving $1 \Rightarrow 2$. Since μ_i is countably additive, it is continuous from below. Recall the definition of A_1, B_n from above; we have

$$F_i^t(r_0) = \mu_i(A^t(r_0)) = \mu_i(A_1) = \lim_{n \to \infty} \mu_i(B_n),$$

where the second equality follows from the regularity of μ_i. Since individual 1 is more risk averse in terms of RCR than individual 2 at t, it follows that $\mu_1(B_n) \le \mu_2(B_n)$ for all n, hence that $F_1^t(r_0) \le F_2^t(r_0)$.

We proceed to show $2 \Rightarrow 1$. Take any $D \in \mathfrak{D}_t$ and consider $N(D, t)$. Let

$$r_0 = \min\{r(u,t) | u \in N(D,t)\},$$

hence $N(D,t) = \{u \in \mathbf{U} | r(u,t) \ge r_0\}$. We have therefore

$$\rho_i^D(t) = \mu_i(N(D,t) = 1 - F_i^t(r_0),$$

which, along with individual 1 being more risk averse in terms of RUF than individual 2 at t, implies $\rho_1^D(t) \ge \rho_2^D(t)$.

6 Proofs

All the proofs of the results stated in the previous sections depend in one way or another on the structure of the algebra $\mathfrak{U}$. We therefore begin with a study of $\mathfrak{U}$.

6.1 Structure of the Algebra $\mathfrak{U}$

6.1.1 Statement on the Structure

To describe the structure of $\mathfrak{U}$ we need two notions: semiring and relative interior. Let $\mathbf{S}$ be an arbitrary set; a class $\mathscr{A}$ of subsets of $\mathbf{S}$ is said to be a semiring, if it contains the empty set and is closed under finite intersection, and $A, B \in \mathscr{A}$ with $A \subseteq B$ implies the existence of a collection of disjoint set $C_1, \ldots, C_n$ in $\mathscr{A}$ such that $B \backslash A = C_1 \cup \cdots \cup C_n$. For more details we refer to, for instance, Billingsley (1995).

To define the notion of relative interior we let V be a vector space over the real numbers $\mathbb{R}$. At this juncture, it is also appropriate to introduce some concepts that, although irrelevant to the definition of relative interior, will prove useful in a little while. Given $v_1, \ldots, v_n \in V$ and $v = \sum_{i=1}^n \alpha_i v_i$, v is called a linear combination of $v_1, \ldots, v_n$ if every α_i is a real number; a positive combination if every $\alpha_i \geq 0$; an affine combination if $\sum_{i=1}^n \alpha_i = 1$; a convex combination if $\sum_{i=1}^n \alpha_i = 1$ and every $\alpha_i \geq 0$. For any subset A of V, the linear space (affine space, convex cone, convex hull), denoted by $\mathrm{span}(A)$ ($\mathrm{aff}(A)$, $\mathrm{pos}(A)$, $\mathrm{conv}(A)$), generated by A is the set of all linear (affine, positive, convex) combinations of elements in A. Let $\mathrm{ext}A$ be the set of extreme points of $\mathrm{conv}(A)$. A polytope is the convex hull generated by a finite subset of V. A set A is called a cone if $A = \mathrm{pos}(A)$. A subset of V is said to be full-dimensional if it is not contained in any hyperplane of V. The relative interior of A, denoted by $\mathrm{ri}A$, is the interior of A in the relative topology of $\mathrm{aff}(A)$.

We endow $C(\mathbf{I})$ with the weak topology and $\mathbf{U}$ the relative topology. The structure of $\mathfrak{U}$ is described in the following proposition (cf. Gul and Pesendorfer (op. cit., Proposition 6)):

Proposition 1 Let $\mathcal{H} = \{\mathrm{ri}K \mid K \in \mathcal{K}^*\} \cup \{\varnothing\}$. Then (i) $\mathcal{H}$ is a semiring;
(ii) $\mathfrak{U} = \{\cup_{i=1}^n H_i \mid H_i \in \mathcal{H}, i, n = 1, 2, \ldots\}$.

To prove this proposition we shall establish a series of auxiliary lemmas, which are infinite-dimensional analogues of the results in Appendix A of Gul and Pesendorfer (ibid.). The general principle in extending Gul and Pesendorfer's results is the following. Recall that Gul and Pesendorfer (2006a) studied random expected utility theory in the context of the n-dimensional Euclidean space and distinguished between two types of sets: sets of dimension n and sets of dimension less than n. These two types of sets can be seen from a more general viewpoint: full-dimensional sets and non-full-dimensional sets, which applies not only to finite-dimension space but also to infinite-dimensional space. We shall show that Gul and Pesendorfer (2006a)'s arguments still hold true in $C[0, M]$ if one replaces a set of dimension n with a full-dimensional set and a set of dimension less than n with a non-full-dimensional set. But to begin with, let us recount some functional-theoretic preliminaries.

6.1.2 Functional-theoretic Preliminaries

Recall that $C(\mathbf{I})$ is the set of all continuous functions on $\mathbf{I}$; it becomes a Banach space when associated with the supremum norm

$$\|u\| = \max_{x\in[0,M]} |u(t)| \text{ for } u \in C(\mathbf{I}).$$

Let $ca(\mathbf{I})$ be the space of Radon measures on $\mathfrak{B}(\mathbf{I})$. Given u in $C(\mathbf{I})$ and x in $ca(\mathbf{I})$, we define on $C(\mathbf{I}) \times ca(\mathbf{I})$ a bilinear form

$$\langle u, x\rangle = \int_0^M u(t)dx(t). \tag{6.1}$$

Concerning the lower limit of the integral, we remark that to avoid extra symbolism, the symbol, 0, will be used to denote either the origin of $C(\mathbf{I})$, or that of $ca(\mathbf{I})$, or the number zero, and the context is supposed to make clear what it indicates indeed.

As has been seen in Sections 3 and 4, we have to deal with two different sets of utility functions. To distinguish them let

$$\mathbf{U}_1 = \{u \in C(\mathbf{I})|\ u(0) = 0\},\ \mathbf{U}_2 = \{u \in C^1(\mathbf{I})|\ u(0) = 0\}. \tag{6.2}$$

The following notion from Aliprantis and Border (1999, p. 211), is of fundamental importance to the development that follows:

Definition 6.1 A dual pair is a pair (V, V') of vector spaces together with a bilinear functional $(v, v') \to \langle v, v'\rangle$ from $V \times V'$ to $\mathbb{R}$ that separates the points of V and V'. That is,

(i) The mapping $v' \to \langle v, v'\rangle$ is linear for every $v \in V$;
(ii) The mapping $v \to \langle v, v'\rangle$ is linear for every $v' \in V'$;
(iii) If $\langle v, v'\rangle = 0$ for every $v' \in V'$, then $v = 0$;
(iv) If $\langle v, v'\rangle = 0$ for every $v \in V$, then $v' = 0$

According to Aliprantis and Border (op. cit., pp. 211-212), $(C(\mathbf{I}), ca(\mathbf{I}))$ is a dual pair, and by putting on $ca(\mathbf{I})$ the weak* topology, $C(\mathbf{I})$ and $ca(\mathbf{I})$ become weakly dual to each other, and they are both locally convex Hausdorff spaces. This fact can be thought of as an infinite-dimensional analogue of the fact that the finite-dimensional Euclidean space $\mathbb{R}^n$ is self-dual. Now we have to establish a similar fact for $\mathbf{U}_i$, i.e., to find a subset $\tilde{\mathbf{X}}_i$ of $ca(\mathbf{I})$ such that $(\mathbf{U}_i, \tilde{\mathbf{X}}_i)$ is a dual pair.

We shall do this for $\mathbf{U}_2$ only, as a similar yet easier procedure applies also to $\mathbf{U}_1$. To begin with, recall that $\|\cdot\|_v$ denotes the total variation norm on $ca(\mathbf{I})$. We first study the continuity of the bilinear form (6.1) with respect to this norm:

Lemma 6.1 For any $u \in \mathbf{U}_2$, $\langle u, x\rangle$ is continuous in x with respect to the total variation norm.[3]

Proof. To prove this we begin with some preliminaries. Recall that for each $x \in ca(\mathbf{I})$ there exist two positive Radon measures x^+, x^- such that $x = x^+ - x^-$, and any positive Radon measure z induces on $[0, M]$ a real-valued function, that is, its decumulative distribution function:

$$G_z(t) = x((t, M]).$$

[3]This result must be available in the literature, but I fail to find out a reference.

With its aid, we define $G_x = G_{x^+} - G_{x^-}$ for any $x \in ca(\mathbf{I})$. Let $x_n \to x$ in the total variation norm, so that $\int_0^M |G_{x_n}(t) - G_x(t)|dt \to 0$ as $n \to \infty$. By Hirsch and Lacombe (1999, Theorem 3.8 and Exercise 14.b) we have

$$\int_0^M u(t)dz(t) = -\int_0^M u(t)dG_z(t) = \int_0^M G_z(t)\dot{u}(t)dt.$$

It follows that

$$\begin{aligned}
|\langle u, x_n\rangle - \langle u, x\rangle| &= \left|\int_0^M (G_{x_n}(t) - G_x(t))\dot{u}(t)dt\right| \\
&\leq \int_0^M |G_{x_n}(t) - G_x(t)|\,|\dot{u}(t)|dt \\
&\leq \|\dot{u}\| \int_0^M |G_{x_n}(t) - G_x(t)|\,dt \to 0.
\end{aligned}$$

This means that $\langle u, x\rangle$ is continuous in x, and the proof is thus completed. ∎

Let $\mathbf{U}_2^\perp = \{x \in ca(\mathbf{I}) |\ \langle u, x\rangle = 0 \text{ for all } u \in \mathbf{U}_2\}$; it is, as a consequence of Lemma 6.1, a closed subspace of $ca(\mathbf{I})$. Let $\tilde{\mathbf{X}}_2 = ca(\mathbf{I})/\mathbf{U}_2^\perp$ be the quotient space (for its definition see Conway (1990, Section III.4, p. 73)). For each $x \in ca(\mathbf{I})$ let $[x] = x + \mathbf{U}_2^\perp$, so that $[x] \in \tilde{\mathbf{X}}_2$. There are several points to be noted here. The first is about the norm on $\tilde{\mathbf{X}}_2$: Set

$$\|[x]\|_v = \inf\{\|x + z\|_v |\ z \in \mathbf{U}_2^\perp\}.$$

Since $\mathbf{U}_2^\perp$ is closed, $\|[x]\|_v$ is, according to Conway (ibid.), a norm on $\tilde{\mathbf{X}}_2$. The second is about the bilinear form on $\tilde{\mathbf{X}}_2$: for $u \in \mathbf{U}_2$ and $x' \in [x]$, we may suppose $x' = x + z$ for some $z \in \mathbf{U}_2^\perp$; then noting that $\langle u, z\rangle = 0$, we have $\langle u, x'\rangle = \langle u, x + z\rangle = \langle u, x\rangle$. Based on this fact we can unambiguously define $\langle u, [x]\rangle = \langle u, x\rangle$ for all $[x] \in \tilde{\mathbf{X}}_2$. For its continuity we claim that

LEMMA 6.2 For any $u \in \mathbf{U}_2$, $\langle u, [x]\rangle$ is continuous with respect to $\|[\cdot]\|_v$.

Proof. Suppose $\|[x_n]\|_v \to \|[x]\|_v$. This means $\|[x_n - x]\|_v \to 0$. Since $\mathbf{U}_2^\perp$ is closed, there exists a $z_n \in \mathbf{U}_2^\perp$ such that $\|[x_n - x]\|_v = \|x_n - x - z_n\|_v$. Then noting that $\langle u, z_n\rangle = 0$, we have using Lemma 6.1,

$$\langle u, [x_n]\rangle - \langle u, [x]\rangle = \langle u, x_n - x\rangle = \langle u, x_n - x - z_n\rangle \to 0.$$

This completes the proof. ∎

The third point is about $\mathbf{X}$. For any $x, y \in \mathbf{X}$ with $x \neq y$, it is easy to deduce from the definition of $\mathbf{X}$ that $[x] \neq [y]$, and so by identifying every $x \in \mathbf{X}$ with $[x]$ we arrive at the inclusion: $\mathbf{X} \subseteq \tilde{\mathbf{X}}_2$. The last point is about notation: to reduce cumbersome notation we will write x for $[x]$ and suppose $x \notin \mathbf{U}_2^\perp$ unless $x = 0$; according to this, we can write, for example, $\langle u, x\rangle$ for $\langle u, [x]\rangle$ and $\|x\|_v$ for $\|[x]\|_v$.

LEMMA 6.3 The triplet, $(\mathbf{U}_2, \tilde{\mathbf{X}}_2, \langle,\rangle)$, is a dual pair.

Proof. It is obvious that both $\mathbf{U}_2$ and $\tilde{\mathbf{X}}_2$ are vector spaces, and so it remains to show that $\langle u, x\rangle = 0$ for all $x \in \tilde{\mathbf{X}}_2$ implies $u = 0$, and that $\langle u, x\rangle = 0$ for all $u \in \mathbf{U}_2$ implies $x = 0$. But note that $\langle u, x\rangle = 0$ for all $x \in \tilde{\mathbf{X}}_2$ implies $\langle u, y\rangle = 0$ for all $y \in ca(\mathbf{I})$, and so the former is an immediate consequence of Corollary 1.2 in IV, §1 of Lang (1993). For the latter, $\langle u, x\rangle = 0$ for all $u \in \mathbf{U}_2$ implies $x \in \mathbf{U}_2^{\perp}$, and therefore $x \in [0]$, which in turn, by our notational convention above, means that $x = 0$. This completes the proof. ■

Then by putting on $\tilde{\mathbf{X}}_2$ the weak* topology, we have that $\mathbf{U}_2$ and $\tilde{\mathbf{X}}_2$ are weakly dual to each other, and they are both locally convex Hausdorff spaces. Likewise, let $\mathbf{U}_1^{\perp} = \{x \in ca(\mathbf{I}) | \langle u, x\rangle = 0 \text{ for all } u \in \mathbf{U}_1\}$ and $\tilde{\mathbf{X}}_1 = ca(\mathbf{I})/\mathbf{U}_1^{\perp}$. Following the above argument we can establish that $(\mathbf{U}_1, \tilde{\mathbf{X}}_1, \langle,\rangle)$ is a dual pair, and by putting on $\tilde{\mathbf{X}}_1$ the weak* topology, $\mathbf{U}_1$ and $\tilde{\mathbf{X}}_1$ become weakly dual to each other and they are both locally convex Hausdorff spaces. We conclude this subsection with a notational convention: since the following development holds valid both for $(\mathbf{U}_1, \tilde{\mathbf{X}}_1)$ and for $(\mathbf{U}_2, \tilde{\mathbf{X}}_2)$, we shall henceforth use $(\mathbf{U}, \tilde{\mathbf{X}})$ to denote either of them.

6.1.3 Lemmas

We begin with the introduction of some further concepts; a reference for them is to Aliprantis and Border (ibid., p. 197). Given $u \in \mathbf{U}$, $\mathbf{x} = (x_1, \ldots, x_m)$ with each x_i in $\tilde{\mathbf{X}}$ and m an integer, we define a vector

$$\langle u, \mathbf{x}\rangle = (\langle u, x_1\rangle, \ldots, \langle u, x_m\rangle).$$

For a real number α (in particular, $\alpha = 0$), by $\langle u, \mathbf{x}\rangle \leq \alpha$ we shall mean that $\langle u, x_i\rangle \leq \alpha$ for all i, and $\langle u, \mathbf{x}\rangle \ll \alpha$ that $\langle u, x_i\rangle < \alpha$ for all i, and $\langle u, \mathbf{x}\rangle = \alpha$ that $\langle u, x_i\rangle = \alpha$ for all i. For some $x \neq 0$ in $\tilde{\mathbf{X}}$ and some real number α, a set of the form $[x \leq \alpha] = \{u \in \mathbf{U} | \langle u, x\rangle \leq \alpha\}$ is called a weak half space in $\mathbf{U}$, and $[x < \alpha] = \{u \in \mathbf{U} | \langle u, x\rangle < \alpha\}$ a strict half space; similarly for $[x \geq \alpha]$ and $[x > \alpha]$. A hyperplane, $[x = \alpha]$, is the intersection of the two weak half spaces, $[x \leq \alpha]$ and $[x \geq \alpha]$. A polyhedron is the intersection of finitely many weak half spaces.

A subset K of $\mathbf{U}$ is said to be a polyhedral cone if it is both a polyhedron and a cone. Let $\mathcal{K}$ be the set of all polyhedral cones of $\mathbf{U}$. It is easily seen that a polyhedral cone must be of the form

$$K(\mathbf{x}; \mathbf{y}) = \{u \in \mathbf{U} | \langle u, \mathbf{x}\rangle = 0, \langle u, \mathbf{y}\rangle \leq 0\}, \tag{6.3}$$

where $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_n)$, with all x_i, y_i in $\tilde{\mathbf{X}}$. Here we make a convention that in Eq. (6.3), if $m = 0$, there are inequalities alone, i.e., $\langle u, \mathbf{y}\rangle \leq 0$; and if $n = 0$, then there are equalities alone, i.e., $\langle u, \mathbf{x}\rangle = 0$.

The equality and inequality constraints in Eq. (6.3), or the hyperplanes and weak half spaces, will play a distinct role in the present study, and therefore it is useful to make a distinction between them. Motivated by this consideration we propose the following

Definition 6.2 Eq. (6.3) is said to be the canonical form of $K(\mathbf{x}; \mathbf{y})$ if $K(\mathbf{x}; \mathbf{y}) \cap [y_i = 0]$ is a proper subset of $K(\mathbf{x}; \mathbf{y})$ for $i = 1, \ldots, n$.

This definition means that if $K(\mathbf{x}; \mathbf{y})$ is of canonical form, then there exists a $u \in K(\mathbf{x}; \mathbf{y})$ such that $\langle u, y_i\rangle < 0$ for some y_i. We make the convention that unless otherwise stated, by writing $K(\mathbf{x}; \mathbf{y})$ we shall mean implicitly that it is of canonical form.

Let $K = K(\mathbf{x}; \mathbf{y})$ with $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_n)$. Let $\mathfrak{I} = \{1, \ldots, n\}$ and $\mathcal{P}(\mathfrak{I})$ the power set of $\mathfrak{I}$; for any $\mathfrak{I}_1$ in $\mathcal{P}(\mathfrak{I})$ let $\mathfrak{I}_1^c$ be the complement of $\mathfrak{I}_1$ in $\mathfrak{I}$. We define $F(K)$ to be a subset of K of the form

$$F(K) = \{u \in K|\ \langle u, \mathbf{y}_1\rangle = 0, \langle u, \mathbf{y}_1^c\rangle \leq 0\},$$

where $\mathbf{y}_1 = (y_i)_{i\in\mathfrak{I}_1}$ and $\mathbf{y}_1^c = (y_i)_{i\in\mathfrak{I}_1^c}$ for some $\mathfrak{I}_1$ in $\mathcal{P}(\mathfrak{I})$. It is obvious that $F(K)$ is also a polyhedral cone, and it is customarily called in convex analysis a face of K. Let $\mathfrak{F}(K)$ be the set of all such $F(K)$'s, and note that by taking $\mathfrak{I}_1 = \varnothing$, we have $K \in \mathfrak{F}(K)$.

Likewise, a polyhedral cone in $\tilde{\mathbf{X}}$ is a set of the form

$$\{x \in \tilde{\mathbf{X}}|\ \langle \mathbf{u}, x\rangle \leq 0\}$$

for some $\mathbf{u} = (u_1, \ldots, u_n)$ with u_i in $\mathbf{U}$. We shall not discuss its canonical form here, simply because it is irrelevant to the following investigation. Let L be a convex subset of $\tilde{\mathbf{X}}$; its normal cone at $x_0 \in L$ in $\mathbf{U}$, denoted by $N(L, x_0)$, is defined by

$$N(L, x_0) = \{u \in \mathbf{U}|\ \langle u, x\rangle \leq \langle u, x_0\rangle \text{ for all } x \in L\}.$$

For $D \in \mathfrak{D}$ and $x \in D$, by $N(D, x)$ we shall mean $N(\mathrm{conv}(D), x)$. Let $\mathcal{K}^* = \{N(D, x)|\ D \in \mathfrak{D}, x \in D\}$, i.e., $\mathcal{K}^*$ is the set of normal cones of D at x for all $x \in D$ and $D \in \mathfrak{D}$. It is obvious that $N(D, x) = N(D - \{x\}, 0)$ and $\mathcal{K}^* \subseteq \mathcal{K}$.

Lemma 6.4 For a polyhedral cone $K(\mathbf{x}; \mathbf{y}) \in \mathcal{K}$ its relative interior is given by

$$\mathrm{ri}K(\mathbf{x}; \mathbf{y}) = \{u \in K(\mathbf{x}; \mathbf{y})|\ \langle u, \mathbf{y}\rangle \ll 0\}.$$

Proof. Let $K = K(\mathbf{x}; \mathbf{y})$ and $\mathbf{y} = (y_1, \ldots, y_n)$. We begin with the simplest case of $n = 0$. In this case K itself is a linear subspace, so that $K = \mathrm{aff}K$, and hence $K = \mathrm{ri}K$.

Now suppose $n \geq 1$. Let $A = \{u \in \mathbf{U}|\ \langle u, \mathbf{x}\rangle = 0, \langle u, \mathbf{y}\rangle \ll 0\}$. We first show that A is nonempty. For this let $X = \mathrm{span}(\{x_1, \ldots, x_m\})$ and $Y = \mathrm{conv}(\{y_1, \ldots, y_n\})$; we claim that $X \cap Y = \varnothing$. For otherwise we would have

$$\sum_{i=1}^{m} \alpha_i x_i = \sum_{j=1}^{n} \lambda_j y_j,\ \alpha_i \in \mathbb{R}, \lambda_j \geq 0, \sum_j \lambda_j = 1.$$

Since at least one of λ_j's is non-vanishing, we assume for definiteness that $\lambda_n \neq 0$; there then follows

$$y_1 = \sum_{i=1}^{m} \alpha_i x_i (\text{ if } n = 1),\ \text{or}, \lambda_n y_n = \sum_{i=1}^{m} \alpha_i x_i - \sum_{j=1}^{n-1} \lambda_j y_j (\text{ if } n \geq 2).$$

Recall that $\langle u, \mathbf{x}\rangle = 0$ and $\langle u, \mathbf{y}\rangle \leq 0$ for all $u \in K(\mathbf{x}; \mathbf{y})$, and so $\langle u, y_n\rangle = 0$ for all $u \in K(\mathbf{x}; \mathbf{y})$. But this contradicts that $K(\mathbf{x}; \mathbf{y})$ is of canonical form, hence $X \cap Y = \varnothing$. Since $\tilde{\mathbf{X}}$ is Hausdorff, it follows from Aliprantis and Border (ibid., Corollaries 5.22 and 5.30) that X is closed and Y is compact, hence from Aliprantis and Border (ibid., Theorem 5.79), that there exists a continuous linear functional, u, such that

$$\langle u, x\rangle > \langle u, y\rangle,\ \text{for all } x \in X, y \in Y. \tag{6.4}$$

As Y is compact we have $\langle u, y\rangle \geq \alpha$ for all $y \in Y$ and some real number α. This along with the inequality (6.4) and the fact that X is a linear space, implies that $u(x) = 0$ for all $x \in X$, so that $\langle u, \mathbf{x}\rangle = 0$ and $\langle u, \mathbf{y}\rangle \ll 0$. Furthermore by Aliprantis and Border (ibid., Theorem 5.93) we have $u \in \mathbf{U}$, hence $u \in A$. This proves that A is nonempty.

Let $\mathbf{U}_0 = \{u \in \mathbf{U}|\ \langle u, \mathbf{x}\rangle = 0\}$. We next show that $\mathrm{aff}K = \mathbf{U}_0$. Since $0 \in K$, it follows that aff$K$ must be a linear space. Noticing that $\mathbf{U}_0$ is a linear subspace of $\mathbf{U}$ that contains K, we have aff$K \subseteq \mathbf{U}_0$, as aff(K) is the smallest affine space containing K. On the other hand, let $B(u_0, \epsilon) = \{u \in \mathbf{U} | \ \|u - u_0\| < \epsilon\}$, where $\| \cdot \|$ denotes the supremum norm on $\mathbf{U}$. Since A is nonempty, it follows that there exists an $\epsilon_0 > 0$ and a $u_0 \in A$ such that $\varnothing \neq B(u_0, \epsilon_0) \cap \mathbf{U}_0 \subseteq K$. It is not hard to see that aff$(B(u_0, \epsilon_0) \cap \mathbf{U}_0) = \mathbf{U}_0$, and so $\mathbf{U}_0 \subseteq$ aff(K), hence $\mathbf{U}_0 =$ aff(K).

From this result we can deduce at once that $A \subseteq$ ri$K$. It remains therefore to show the converse: ri$K \subseteq A$. Suppose by way of contradiction that there exists a $u_1 \in$ ri$K \backslash A$. This means that there exist a $\varepsilon_1 > 0$ and an integer i such that $B(u_1, \varepsilon_1) \cap \mathbf{U}_0 \subseteq K$ and $\langle u_1, y_i \rangle = 0$. Since $K(\mathbf{x}; \mathbf{y})$ is of canonical form, it follows that y_i is not a linear combination of $x_1, \ldots, x_m$, and hence that according to Aliprantis and Border (ibid., Corollary 5.92), there exists a u_2 such that $\langle u_2, \mathbf{x} \rangle = 0$ and $\langle u_2, y_i \rangle > 0$. Based on this we can find an $\alpha > 0$ sufficiently small such that $\alpha u_2 + u_1 \in B(u_1, \varepsilon_1) \cap \mathbf{U}_0$ and $\alpha u_2 + u_1 \notin K$; but this contradicts $B(u_1, \varepsilon_1) \cap \mathbf{U}_0 \subseteq K$. So ri$K \subseteq A$ and thus ri$K = A$. This completes the proof. ■

LEMMA 6.5 Let $K_1 = K(\mathbf{x}^1; \mathbf{y}^1)$, $K_2 = K(\mathbf{x}^2; \mathbf{y}^2)$ be two polyhedral cones in $\mathcal{K}$. If ri$K_1 \cap$ri$K_2 \neq \varnothing$, then ri$K_1 \cap$ ri$K_2 =$ ri$(K_1 \cap K_2)$. (Cf. Rockafellar (1970, Theorem 6.5).)

Proof. Let $\mathbf{y}^t = (y_1^t, \ldots, y_{n_t}^t)$, $t = 1, 2$. By Lemma 6.4 we have

$$\text{ri}K_t = \{u \in \mathbf{U} | \ \langle u, \mathbf{x}^t \rangle = 0, \langle u, \mathbf{y}^t \rangle \ll 0\}, t = 1, 2.$$

Since ri$K_1 \cap$ ri$K_2 \neq \varnothing$, it follows that there exists a $u \in K_1 \cap K_2$ but $u \notin [y_i^t = 0]$, for all $i = 1, \ldots, n_t^t$ and $t = 1, 2$. As a reslult, the canonical form of $K_1 \cap K_2$ must be given by

$$\{u \in \mathbf{U} | \ \langle u, \mathbf{x} \rangle = 0, \langle u, \mathbf{y} \rangle \leq 0\},$$

where $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2)$ and $\mathbf{y} = (\mathbf{y}^1, \mathbf{y}^2)$. We then have according to Lemma 6.4,

$$\text{ri}(K_1 \cap K_2) = \{u \in \mathbf{U} | \ \langle u, \mathbf{x} \rangle = 0, \langle u, \mathbf{y} \rangle \ll 0\},$$

and therefore ri$K_1 \cap$ ri$K_2 =$ ri$(K_1 \cap K_2)$. ■

LEMMA 6.6 For any $K \in \mathcal{K}$, we have (cf. Gul and Pesendorfer (op. cit., Proposition 5)):

(i) ri$F_1 \cap$ ri$F_2 = \varnothing$ for any $F_1, F_2 \in \mathfrak{F}(K)$ with $F_1 \neq F_2$.
(ii) $K = \bigcup_{F \in \mathfrak{F}(K)} \text{ri}F$.

Proof. To see statement (i) we let $K = K(\mathbf{x}, \mathbf{y})$ with $\mathbf{y} = (y_1, \ldots, y_n)$, and take $\mathfrak{I}_1, \mathfrak{I}_2 \in \mathcal{P}(\mathfrak{I})$ such that

$$F_k = \{u \in K | \ \langle u, \mathbf{y}_k \rangle = 0, \langle u, \mathbf{y}_k^c \rangle \leq 0\}, k = 1, 2,$$

where $\mathbf{y}_k = (y_i)_{i \in \mathfrak{I}_k}$ and $\mathbf{y}_k^c = (y_i)_{i \in \mathfrak{I}_k^c}$. Since $F_1 \neq F_2$, it follows that $\mathfrak{I}_1 \neq \mathfrak{I}_2$, hence that there exists an $i \in \mathfrak{I}$ such that either $i \in \mathfrak{I}_1 \cap \mathfrak{I}_2^c$ or $i \in \mathfrak{I}_1^c \cap \mathfrak{I}_2$. Since the two cases can be treated likewise let us take the first; in which we have according to Lemma 6.4,

$$\text{ri}F_1 \subseteq \{u \in \mathbf{U} | \ \langle u, y_i \rangle = 0\},$$
$$\text{ri}F_2 \subseteq \{u \in \mathbf{U} | \ \langle u, y_i \rangle < 0\},$$

whence it follows that ri$F_1 \cap$ ri$F_2 = \varnothing$. This completes the proof of statement (i).

As regards statement (ii), it is evident that

$$\bigcup_{F \in \mathfrak{F}(K)} \text{ri}F \subseteq K.$$

To show the converse we take $u \in K$. By the definition of K there must exist an $\mathfrak{I}_1 \in \mathcal{P}(\mathfrak{I})$, which may possibly be empty or just $\mathfrak{I}$ itself, such that $\langle u, \mathbf{y}_1 \rangle = 0$ and $\langle u, \mathbf{y}_1^c \rangle \ll 0$, where $\mathbf{y}_1 = (y_i)_{i \in \mathfrak{I}_1}$ and $\mathbf{y}_1^c = (y_i)_{i \in \mathfrak{I}_1^c}$. Let

$$F = \{u \in K | \ \langle u, \mathbf{y}_1 \rangle = 0, \langle u, \mathbf{y}_1^c \rangle \leq 0\};$$

then $u \in \mathrm{ri} F$. So $K \subseteq \bigcup_{F \in \mathfrak{F}(K)} \mathrm{ri} F$, and this completes the proof of statement (ii). ■

The previous lemmas have studied the properties of $\mathcal{K}$; the next one specializes to $\mathcal{K}^*$ (cf. Gul and Pesendorfer (ibid., Propositions 1 and 3)):

Lemma 6.7 (i) The set $\mathcal{K}^*$ is closed under finite intersection.
(ii) For any $K \in \mathcal{K}^*$, we have $F(K) \in \mathcal{K}^*$.
(iii) $N(D, x) \in \mathcal{K}^*$ is of full dimension if and only if $x \in \mathrm{ext} D$.

Proof. The validity of statement (i) is an immediate consequence of the observation that for $N(D_i, z_i) \in \mathcal{K}^*$

$$\bigcap_{i=1}^{m} N(D_i, z_i) = N(\sum_{i=1}^{m} \frac{1}{m} D_i, \sum_{i=1}^{m} \frac{1}{m} z_i).$$

To see statement (ii) let $K = N(D, z) \in \mathcal{K}^*$ with its canonical form given by $K(\mathbf{x} - z, \mathbf{y} - z)$, where $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ and all $x_i, y_j \in D \subseteq \mathbf{X}$. Let $F(K) = \{u \in K | \ \langle u, \mathbf{y}_1 - z \rangle = 0, \langle u, \mathbf{y}_1^c - z \rangle \leq 0\}$, where $\mathbf{y}_1 = (y_i)_{i \in \mathfrak{I}_1}$ and $\mathbf{y}_1^c = (y_i)_{i \in \mathfrak{I}_1^c}$ for some $\mathfrak{I}_1$ in $\mathcal{P}(\mathfrak{I})$. To show $F(K) \in \mathcal{K}^*$, we assume for definiteness and without loss of generality that $\mathfrak{I}_1 = \{1, \ldots, k\}$ and $\mathfrak{I}_1^c = \{k+1, \ldots, n\}$. Let $\bar{z} = \frac{1}{k+1}(y_1 + \cdots + y_k + z)$, so that $\bar{z} \in \mathbf{X}$. Let $D_1 = \{y_1, \ldots, y_k, z, \bar{z}\}$, $D_2 = \{y_{k+1}, \ldots, y_n, z\}$, so that $D_1, D_2 \in \mathfrak{D}$ and $N(D_1, \bar{z}) = \{u \in \mathbf{U} | \ \langle u, \mathbf{y}_1 - z \rangle = 0\}$, $N(D_2, z) = \{u \in \mathbf{U} | \ \langle u, \mathbf{y}_1^c - z \rangle \leq 0\}$. Hence we get $F(K) = N(D, z) \cap N(D_1, \bar{z}) \cap N(D_2, z)$. From statement (i) we have $F(K) \in \mathcal{K}^*$.

To see statement (iii) we assume in the first place that $x \in \mathrm{ext} D$. It then follows from Aliprantis and Border (op. cit., Corollary 7.90) that x is a strongly exposed point of the polytope $\mathrm{conv}(D)$, which means that $\langle u, z \rangle < \langle u, x \rangle$ for some $u \in \mathbf{U}$ and all $z \in D \backslash \{x\}$. So there exists a neighborhood of u that is contained in $N(D, x)$. This implies that $N(D, x)$ has a nonempty interior, hence is full-dimensional.

We assume in the second place that $N(D, x)$ is of full dimension, and, by way of contradiction, that $x \notin \mathrm{ext} D$. This means that x is a convex combination of some points in D, say $z_1, \ldots, z_n$, and consequently $N(D, x)$ must be contained in the hyperplane $[x - z_1 = 0]$. But this contradicts that $N(D, x)$ is of full dimension. The proof is thus completed. ■

6.1.4 Proof of Proposition 1

All the preliminary machinery having been developed, the proof of Proposition 1 will proceed quite swiftly. For statement (i) we take $K_t = N(D_t, z_t) \in \mathcal{K}^*$, $t = 1, 2$; let $H_1 = \mathrm{ri} K_1$, $H_2 = \mathrm{ri} K_2$. We first show that $H_1 \cap H_2 \in \mathcal{H}$, which is trivially true if $H_1 \cap H_2 = \varnothing$. So suppose that $H_1 \cap H_2 \neq \varnothing$. Then, in view of Lemma 6.5, $H_1 \cap H_2 = \mathrm{ri}(K_1 \cap K_2)$; from statement (i) of Lemma 6.7 we have $K_1 \cap K_2 \in \mathcal{K}^*$, hence $H_1 \cap H_2 \in \mathcal{H}$.

We next show that $H_2 \backslash H_1$ is the union of a collection of disjoint sets in $\mathcal{H}$. To see this let the canonical forms of K_t, $t = 1, 2$ be given by

$$\{u \in \mathbf{U} | \ \langle u, \mathbf{x}^t - z_t \rangle = 0, \langle u, \mathbf{y}^t - z_t \rangle \leq 0\},$$

where $\mathbf{x}^t = (x_1^t, \ldots, x_{m_t}^t)$ and $\mathbf{y}^t = (y_1^t, \ldots, y_{n_t}^t)$. We have by reference to Lemma 6.4,

$$H_t = \{u \in \mathbf{U} | \ \langle u, \mathbf{x}^t - z_t \rangle = 0, \langle u, \mathbf{y}^t - z_t \rangle \ll 0\}.$$

The complement, H_1^c, of H_1 is then the union of a collection of disjoint sets of the following forms

$$\begin{aligned} A_0 =& \{u \in \mathbf{U} | \ \langle u, x_1^1 - z_1 \rangle \neq 0\}, A_k = \{u \in \mathbf{U} | \ \langle u, \mathbf{x}_k^1 - z_1 \rangle = 0, \langle u, \bar{z}_k \rangle < 0\}, \text{ or} \\ B_0 =& \{u \in \mathbf{U} | \ \langle u, \mathbf{x}^1 - z_1 \rangle = 0, \langle u, z_1 - y_1^1 \rangle \leq 0\}, \\ B_l =& \{u \in \mathbf{U} | \ \langle u, \mathbf{x}^1 - z_1 \rangle = 0, \langle u, \mathbf{y}_l^1 - z_1 \rangle \ll 0, \langle u, z_1 - y_{l+1}^1 \rangle \leq 0\}, \end{aligned}$$

where $\mathbf{x}_1^1 = (x_1^1, \ldots, x_k^1)$, $\bar{z}_k = \pm(x_{k+1}^1 - z_1)$, $k = 1, \ldots, m_1 - 1$, and $\mathbf{y}_1^1 = (y_1^1, \ldots, y_l^1)$, $l = 1, \ldots, n_1 - 1$.

Let us take B_l, $l = 1, \ldots, n_1 - 1$, for example; a similar but easier argument holds for all other sets. Let

$$\begin{aligned} B_{l1} =& \{u \in B_l | \ \langle u, z_1 - y_{l+1}^1 \rangle < 0\}, \\ B_{l2} =& \{u \in B_l | \ \langle u, z_1 - y_{l+1}^1 \rangle = 0\}, \end{aligned}$$

and therefore $B_l = B_{l1} \cup B_{l2}$. If B_{li} is empty, then $B_{li} \cap H_2 \in \mathcal{H}$, $i = 1, 2$.

Now suppose that neither of B_{li} is empty. For B_{l1}, let $K_{11} = N(E_{11}, z_1)$, $K_{12} = N(E_{12}, y_{l+1}^1)$, $K_{13} = N(E_{13}, \tilde{z}_1)$, where $E_{11} = \{y_1^1, \ldots, y_l^1, z_1\}$, $E_{12} = \{y_{l+1}^1, z_1\}$, $E_{13} = \{x_1^1, \ldots, x_{m_1}^1, z_1, \tilde{z}_1\}$, and $\tilde{z}_1 = \frac{1}{m_1+1}(x_1^1 + \cdots + x_{m_1}^1 + z_1)$. Then we have $K_{11}, K_{12}, K_{13} \in \mathcal{K}^*$, and $B_{l1} = \mathrm{ri}(K_{11} \cap K_{12} \cap K_{13})$. If $B_{l1} \cap H_2 = \varnothing$, then $B_{l1} \cap H_2 \in \mathcal{H}$; otherwise $B_{l1} \cap H_2 = \mathrm{ri}(K_{11} \cap K_{12} \cap K_{13} \cap K_2) \in \mathcal{H}$.

Similarly, for B_{l2}, let $K_{21} = K_{11}$, $K_{22} = N(E_{22}, \tilde{z}_2)$, $K_{23} = K_{13}$, where $E_{22} = \{y_{l+1}^1, z_1, \tilde{z}_2\}$, $\tilde{z}_2 = \frac{1}{2}y_{l+1}^1 + \frac{1}{2}z_1$. Then we have $K_{21}, K_{22}, K_{23} \in \mathcal{K}^*$, and $B_{l2} = \mathrm{ri}(K_{21} \cap K_{22} \cap K_{23})$. If if $B_{l2} \cap H_2 = \varnothing$, then $B_{l2} \cap H_2 \in \mathcal{H}$; otherwise $B_{l1} \cap H_2 = \mathrm{ri}(K_{21} \cap K_{22} \cap K_{23} \cap K_2) \in \mathcal{H}$. Applying the above reasoning to all of A_0, A_k, B_0, we are then able to conclude that $H_2 \backslash H_1 = H_1 \cap H_1^c$ is the union of a collection of disjoint sets in $\mathcal{H}$. This completes the proof of statement (i).

As regards statement (ii) we first show that $\mathfrak{U}$ is an algebra. For this it suffices, as $\mathcal{H}$ is a semiring, to show that $\mathbf{U} \in \mathfrak{U}$. Take two distinct elements $x_1, x_2 \in \mathbf{X}$, and set $K_1 = [x_1 - x_2 \leq 0]$, $K_2 = [x_1 - x_2 \geq 0]$, $K_3 = [x_1 - x_2 = 0]$. It is obvious that $K_1, K_2, K_3 \in \mathcal{K}^*$. We then have according to Lemma 6.4, $\mathrm{ri}K_1 = [x_1 - x_2 < 0]$, $\mathrm{ri}K_2 = [x_1 - x_2 > 0]$, $\mathrm{ri}K_3 = [x_1 - x_2 = 0]$, so that $\mathbf{U} = \mathrm{ri}K_1 \cup \mathrm{ri}K_2 \cup \mathrm{ri}K_3$, hence $\mathbf{U} \in \mathfrak{U}$. And secondly, we show that $\mathcal{K}^* \subseteq \mathfrak{U}$. But this is simply a joint consequence of the second statements of Lemmas 6.6 and 6.7. Finally we show that $\mathfrak{U}$ is the smallest algebra containing $\mathcal{H}$; for this we refer the reader to the concluding paragraph of the proof of Gul and Pesendorfer (op. cit., Proposition 6). This completes the proof of statement (ii).

6.2 Proof of Theorem 2.1

The proof is essentially the same as that of Theorem 1 of Gul and Pesendorfer (2006a); but we shall detail it here for the sake of completeness.

We begin with the first statement: every regular RUF has a unique RCR as its maximizer. To see this let μ be a regular RUF, and we define for every $D \in \mathfrak{D}$ and every $x \in D$:

$$\rho^D(x) = \mu(N(D, x)).$$

It is easily seen that if ρ constitutes a RCR, then it must be the unique one that can maximize μ. Therefore it remains to show that ρ is a RCR. For this we have but to check

$$\sum_{x \in D} \rho^D(x) = 1.$$

Appealing to the regularity of μ, we have $\rho^D(x) = \mu(N^+(D, x))$; since $N^+(D, x) \cap N^+(D, y) = \varnothing$ for any distinct x, y in D, it follows that

$$\sum_{x \in D} \rho^D(x) = \sum_{x \in D} \mu(N^+(D, x)) = \mu(\cup_{x \in D} N^+(D, x) = 1.$$

This indicates that ρ is indeed a RCR, and so the proof of the first statement is completed.

We turn now to the second statement: every RCR can maximize at most one regular RUF. Let ρ be a RCR and μ, μ' be two regular RUF's that are maximized at ρ. We have to show $\mu = \mu'$, that is, $\mu(A) = \mu'(A)$ for all $A \in \mathfrak{U}$. According to Proposition 1, there exists for every $A \in \mathfrak{U}$ a finite number of $H_i \in \mathcal{H}$, $i = 1, \ldots, n$, such that $A = \cup_{i=1}^n H_i$, so it suffices to show that

$$\mu(\cup_{i=1}^n H_i) = \mu'(\cup_{i=1}^n H_i) \text{ for all } H_i \in \mathcal{H}, n = 1, 2, \ldots. \tag{6.5}$$

Let us prove this by induction on n. When $n = 1$, if $H_1 = \varnothing$, then $\mu(H_1) = \mu'(H_1) = 0$; otherwise we have, as $H_1 \in \mathcal{H}$, $H_1 = \mathrm{ri}N(D, x)$ for some $D \in \mathfrak{D}$ and $x \in D$. Since μ, μ' are regular and both maximized at ρ, it follows that $\mu(H_1) = \mu'(H_1) = \rho^D(x)$. This completes the proof of the base case.

For the inductive step, suppose that equation (6.5) holds true for $n \leq k$, and we have to show that it also holds for $n = k + 1$. For notational convenience let $B = \cup_{i=1}^k H_i$, $C = B \cap H_{k+1}$. By the induction hypothesis we have $\mu(B) = \mu'(B)$ and $\mu(H_{k+1}) = \mu'(H_{k+1})$. Moreover, since

$$C = \cup_{i=1}^k (H_i \cap H_{k+1}),$$

and since $\mathcal{H}$ is a semiring, it follows that $H_i \cap H_{k+1} \in \mathcal{H}$ for all $i = 1, \ldots, k$, hence, again by the induction hypothesis, that $\mu(C) = \mu'(C)$. This implies that $\mu(B \backslash C) = \mu'(B \backslash C)$, $\mu(H_{k+1} \backslash C) = \mu'(H_{k+1} \backslash C)$. By noting that $\mu(\cup_{i=1}^{k+1} H_i) = \mu(B \backslash C) + \mu(H_{k+1} \backslash C) + \mu(C)$, the inductive step is completed, hence also the proof of the second statement.

6.3 Proof of Theorem 3.1

We begin with the sufficiency part. Let ρ be a RCR which maximizes a regular RUF μ; we have to show that ρ is mixture continuous, monotone, linear, and extreme. The proof is essentially the same as that of Theorem 2 of Gul and Pesendorfer (2006a), with the checking of the last three properties exactly the same as the latter, and so we shall here prove but the mixture continuity of ρ.

To this end take $D, D' \in \mathfrak{D}$ and $\alpha_n, \alpha \in [0, 1]$ with $\alpha_n \to \alpha$. Recall that Π is endowed with the topology of weak convergence, and so it suffices to show

$$\rho(\alpha_n D + (1 - \alpha_n) D') \to \rho(\alpha D + (1 - \alpha) D') \text{ in the weak topology.} \tag{6.6}$$

When $\alpha \in (0, 1)$, we have for n sufficiently large: $N(\alpha_n D + (1 - \alpha_n) D', \alpha_n x + (1 - \alpha_n) x') =$

$N(D, x) \cap N(D', x')$, which means that $\rho(\alpha_n D + (1-\alpha_n)D')$ is constant, hence the validity of statement (6.6).

For $\alpha = 0$ or 1, the proofs of the two cases are alike, and therefore we shall prove the case $\alpha = 0$ only. Take any $x' \in D'$ and take $\epsilon > 0$ small enough such that $B_\epsilon(x') \cap D' = \{x'\}$. We split $\{\alpha_n\}$ into two subsequences: $\{\alpha_n | \alpha_n > 0\}$ and $\{\alpha_n | \alpha_n = 0\}$, and it is sufficient to show the validity of statement (6.6) for each of these two subsequences. Noting that the statement is trivially true for the second subsequence, it remains to verify its validity for the first one. Without loss of generality we assume $\alpha_n \in (0, 1)$. Since $\alpha_n \to 0$, we can take n to be sufficiently large so that $\alpha_n D + (1-\alpha_n)D' \cap B_\epsilon(x') = \alpha_n D + (1-\alpha_n)\{x'\}$. We have therefore

$$\rho^{\alpha_n D+(1-\alpha_n)D'}(B_\epsilon(x')) = \sum_{x \in D} \mu(N(\alpha_n D + (1-\alpha_n)D', \alpha_n x + (1-\alpha_n)x')).$$

Since $\alpha_n \in (0, 1)$ it follows that $N(\alpha_n D + (1-\alpha_n)D', \alpha_n x + (1-\alpha_n)x') = N(D, x) \cap N(D', x')$. This, together with the regularity of μ, implies that

$$\rho^{\alpha_n D+(1-\alpha_n)D'}(B_\epsilon(x')) = \mu((\bigcup_{x \in D} N(D, x)) \cap N(D', x') = \rho^{D'}(x'),$$

and therefore ρ is mixture continuous.

We turn now to the necessity part. Suppose that ρ is a RCR that is mixture continuous, monotone, linear, and extreme; we intend to show the existence of a regular RUF μ that is maximized at ρ. To this end, we define for each $K = N(D, x) \in \mathcal{K}^*$,

$$\mu(\mathrm{ri}K) = \rho^D(x) \text{ and } \mu(\varnothing) = 0. \tag{6.7}$$

To see that μ is well defined, we first claim that

LEMMA 6.8 Let A be a finite subset of $\tilde{\mathbf{X}}$ and $K = \mathrm{pos}A$. Then $K = N(N(K, 0), 0)$.

Proof. (Cf. Schneider (1993, Theorem 1.6.1).) Let $L = N(K, 0)$. For any x in K, we have $\langle u, x\rangle \leq 0$ for all u in L, and so $x \in N(L, 0)$, hence $K \subseteq N(L, 0)$.

To show the converse we shall show equivalently that $z \notin K$ implies $z \notin N(L, 0)$. For this take $z \notin K$. Since K is weakly* closed (Aliprantis and Border (op. cit., Corollary 5.25)) and the singleton, $\{z\}$, is compact, it follows from Aliprantis and Border (ibid., Theorems 5.79 and 5.93) that there exists a nonzero continuous linear functional $u \in \mathbf{U}$ such that

$$\langle u, x\rangle < \langle u, z\rangle \text{ for all } x \in K.$$

Remembering that K is a cone with $0 \in K$, we must have $\langle u, x\rangle \leq 0$ for all $x \in K$ and $\langle u, z\rangle > 0$. The former means $u \in L$, which, in combination with the latter, implies $z \notin N(L, 0)$. It follows that $N(L, 0) \subseteq K$, hence $K = N(L, 0)$. This completes the proof. ∎

By virtue of this lemma, it is not difficult to see that Gul and Pesendorfer (op. cit., Lemma 1) continues to hold in the current infinite dimensional situation, that is, if ρ is monotone, linear, and extreme, then $x \in D$, $x' \in D'$, and $N(D, x) = N(D', x')$ implies $\rho^D(x) = \rho^{D'}(x')$. It then follows that Eq. (6.7) is well defined.

We first show that μ is a finitely additive probability measure on $\mathfrak{U}$. We begin with an observation. For $K = N(D, x)$, if K is not full-dimensional (hence $\mathrm{int}K = \varnothing$), then by statement (iii) of Lemma 6.7, $x \notin \text{ext}D$, and therefore, by Eq. (6.7) and the fact that ρ is extreme, $\mu(\text{ri}K) = \mu(\text{int}K) = 0$; if K is full-dimensional, then $\text{ri}K = \text{int}K$. In short, we have $\mu(\text{ri}K) = \mu(\text{int}K)$ for every $K \in \mathcal{K}^*$. From this observation and the argument of Gul and Pesendorfer (ibid., pp. 140-142), it follows that μ is finitely additive on $\mathfrak{U}$. There remains to show $\mu(\mathbf{U}) = 1$. For this we take, as above, two distinct measures of $\mathbf{X}$, say x_1, x_2, and let $D = \{x_1, x_2, x_3\}$ with $x_3 = \frac{1}{2}x_1 + \frac{1}{2}x_2$. Let $K_i = N(D, x_i)$, so that $K_i \in \mathcal{K}^*$. Referring to Lemma 6.4 we see that $\text{ri}K_i$'s are pairwise disjoint and $\mathbf{U} = \text{ri}K_1 \cup \text{ri}K_2 \cup \text{ri}K_3$. Since μ is finitely additive, we obtain

$$\mu(\mathbf{U}) = \sum_{i=1}^{3} \mu(\text{ri}K_i) = \sum_{i=1}^{3} \rho^D(x_i) = 1.$$

We next show that μ is maximized at ρ. Before turning to this we first present a lemma:

Lemma 6.9 Suppose that ρ and μ are related by Eq. (6.7) and that μ is finitely additive. Then given $x_1, x_2 \in \mathbf{X}$ with $x_1 \neq x_2$, any subset in $\mathfrak{U}$ of the hyperplane $[x_1 - x_2 = 0]$ has μ-measure zero.

Proof. Let $D = \{x_1, x_2, x_3\}$ with x_3 defined as above. Then $[x_1 - x_2 = 0] = N(D, x_3)$. Since x_3 is not an extreme point, we have $\rho^D(x_3) = 0$, hence $\mu(N(D, x_3)) = 0$, i.e. $[x_1 - x_2 = 0]$ is of μ-measure zero. Since μ is finitely additive, any of its subset in $\mathfrak{U}$ must also be of μ-measure zero. ∎

For any $K = N(D, x) \in \mathcal{K}^*$, if $F \in \mathfrak{F}(K)$ but $F \neq K$, then F must be contained in some hyperplane, and therefore, by Lemma 6.9, is of μ-measure zero. From Lemma 6.6 it then follows that

$$\mu(N(D, x)) = \mu(\text{ri}(N(D, x))) = \rho^D(x), \tag{6.8}$$

hence that μ is maximized at ρ.

Finally we show that μ is regular. For this we need the following (cf. Schneider (op. cit., the remark after Lemma 2.2.3)

Lemma 6.10 For any $D \in \mathfrak{D}$ and $x_1, x_2 \in D$ with $N(D, x_1) \neq N(D, x_2)$, $\text{ri}N(D, x_1) \cap \text{ri}N(D, x_2) = \varnothing$.

Proof. To see this we let $D = \{y_1, \ldots, y_n\}$ (it is assumed implicitly that $x_1 = y_i$ and $x_2 = y_j$ for some $y_i, y_j \in D$). Let $\mathfrak{I}_k \in \mathcal{P}(\mathfrak{I})$, $k = 1, 2$, be such that the canonical form of $N(D, x_k)$ is given by

$$N(D, x_k) = \{u \in \mathbf{U} |\ \langle u, \mathbf{y}_k \rangle = \langle u, x_k \rangle, \langle u, \mathbf{y}_k^c \rangle \leq \langle u, x_k \rangle\}$$

where $\mathbf{y}_k = (y_i)_{i \in \mathfrak{I}_k}$ and $\mathbf{y}_k^c = (y_i)_{i \in \mathfrak{I}_k^c}$. Since $N(D, x_1) \neq N(D, x_2)$, it follows that either $u \in N(D, x_1) \backslash N(D, x_2)$ or $u \in N(D, x_2) \backslash N(D, x_1)$ for some $u \in \mathbf{U}$.

Let us take the former case; a similar argument holds also for the latter case. Noting that $u \in N(D, x_1) \backslash N(D, x_2)$ implies $\langle u, x_1 \rangle > \langle u, x_2 \rangle$, we have $x_2 \neq y_i$ for all $i \in \mathfrak{I}_1$. Hence according to Lemma 6.4,

$$\text{ri}N(D, x_1) \subseteq [x_2 - x_1 < 0];$$

observing that the intersection with $N(D, x_2)$ of the right-hand member is empty, we get $\text{ri}N(D, x_1) \cap \text{ri}N(D, x_2) = \varnothing$. This completes the proof. ∎

Note that if $x \in \mathrm{ext}D$, then $N^+(D,x) = \mathrm{ri}(N(D,x))$; if $x \notin \mathrm{ext}D$, then $N^+(D,x) = \varnothing$. Note also that for any $x_1, x_2 \in \mathrm{ext}D$ with $x_1 \neq x_2$, $N(D,x_1) \neq N(D,x_2)$, hence $\mathrm{ri}N(D,x_1) \cap \mathrm{ri}N(D,x_2) = \varnothing$ by Lemma 6.10. We have therefore

$$\mu(\cup_{x\in D} N^+(D,x)) = \sum_{x\in \mathrm{ext}D} \mu(\mathrm{ri}(N(D,x))) = \sum_{x\in\mathrm{ext}D} \rho^D(x) = 1.$$

This completes the whole proof.

6.4 Proof of Theorem 4.1

Recall that for Theorem 4.1, we take $\mathbf{U} = \{u \in C^1(\mathbf{I}) |\ u(0) = 0\}$. Again we begin by proving the sufficiency of the conditions. The following lemma is a generalization of Gul and Pesendorfer (op. cit., Proposition 7) to the case where the decision problems involved may not be of full dimension. In order to facilitate comparison, the notation of that proposition will be retained as much as is convenient. With this generalization, the proof of the sufficiency of the conditions will proceed in the same way as in Gul and Pesendorfer (ibid., Lemma 7), and therefore will not be presented here.

It is perhaps worthy of mention that the proof of Lemma 7 of Gul and Pesendorfer (2006a) is divided into two cases: D being full-dimensional, in which case the proof depends on their Proposition 7, or non-full-dimensional. The reason for such a division is that their Proposition 7 holds valid only for full-dimensional D. Since Lemma 6.11 below (the counterpart of Gul and Pesendorfer's lemma 7) is shown to be true for any D, either full-dimensional or non-full-dimensional, it is no longer necessary to distinguish between the full-dimensional and non-full-dimensional circumstances in the present context.

Lemma 6.11 Let $D_i \in \mathfrak{D}$, $i = 1, 2, \ldots$, and suppose they converge to D. Let $K = N(D,x)$ for some $x \in D$. Then there exist $K_j \in \mathcal{K}^*$, k_j, and $\varepsilon_j > 0$ for $j = 1, 2, \ldots$ such that (i) $K_{j+1} \subseteq K_j$ for all j; (ii) $\cap_j K_j = K$; (iii) $\cup_{x_i \in D_i \cap B_{\varepsilon_j}(x)} N(D_i, x_i) \subseteq K_j$ for $i > k_j$.

Proof. Since $\mathrm{conv}(D)$ is nonempty, convex, and of finite dimension, it follows from Aliprantis and Border (op. cit., Lemma 7.33) that $\mathrm{ri}(\mathrm{conv}(D)) \neq \varnothing$. Take $y^* \in \mathrm{ri}(\mathrm{conv}(D))$; let $\tilde{D}_j = \{x\} \cup (\frac{j}{j+1}D + \frac{1}{j+1}y^*)$ and $K_j = N(\tilde{D}_j, x)$. The proof of statements (i) and (ii) is the same as that of the first two statements in Gul and Pesendorfer (op. cit., Proposition 7), and so is omitted here.

For statement (iii) it is equivalent to showing that $u \notin K_j$ implies $u \notin \cup_{x_i\in D_i \cap B_{\varepsilon_j}(x)} N(D_i, x_i)$. It follows from statement (ii) that $u \notin K_j$ implies $u \notin K$, and so there exist a $y \in D$ and an $\alpha > 0$ such that $\langle u, y\rangle > \langle u, x\rangle + \alpha$.

Let $B(x,\epsilon) = \{z \in \tilde{\mathbf{X}} |\ \|z - x\|_v < \epsilon\}$. According to Lemma 6.2, $\langle u, z\rangle$ is continuous in z, and so there exists a $\varepsilon_j > 0$ such that $|\langle u, z\rangle| < \alpha/2$ for all $z \in B(0, \varepsilon_j)$. Since the sequence $\{D_i\}$ converges to D, we can choose k_j sufficiently large such that $B(y, \varepsilon_j) \cap D_i \neq \varnothing$ for every $i > k_j$. Take $y_i \in B(y,\varepsilon_j) \cap D_i$; then we have for any $x_i \in B(x, \varepsilon_j) \cap D_i$,

$$\begin{aligned}
&\langle u, y_i\rangle - \langle u, x_i\rangle \\
=&\langle u, y_i\rangle - \langle u, y\rangle + \langle u, y\rangle - \langle u, x\rangle + \langle u, x\rangle - \langle u, x_i\rangle \\
>&-\alpha/2 + \alpha - \alpha/2 = 0;
\end{aligned}$$

hence $u \notin N(D_i, x_i)$ for every $i > k_j$. This completes the proof. ∎

We turn now to the necessity part. Recall that $\dot{u}$ denotes the derivative of u; let $S = \{u \in \mathbf{U} |\ \|u\| + \|\dot{u}\| = 1\}$, so that S is a compact subset of $C(\mathbf{I})$. A subtlety here is that S is not compact in $\mathbf{U}$, which explains why the following lemma seeks to find an open cover of S in $C(\mathbf{I})$, instead of in $\mathbf{U}$. With this lemma, the proof of the necessity of the conditions will proceed in the same way as in Gul and Pesendorfer (ibid., Lemma 6), and therefore will not be presented here.

LEMMA 6.12 Let μ be a RUF that is maximized by a continuous random choice rule ρ. For each $K \in \mathcal{K}^*$ and each $\epsilon > 0$, there exist an open subset O of $C(\mathbf{I})$ and a set $\tilde{K} \in \mathfrak{U}$ such that $K \cap S \subseteq O$, $O \cap \mathbf{U} \subseteq \tilde{K}$, and $\mu(\tilde{K}) - \mu(K) < \epsilon$.

Proof. Let $K = N(D, x)$ for some $D \in \mathfrak{D}$ and $x \in D$; let $\{\delta_i |\ i = 0, 1, \ldots\}$ be the set of all Dirac measures at the rational numbers in $[0, M]$ with δ_0 the Dirac measure at zero. Since the set of rational numbers is dense in $[0, M]$, we have

$$\mathbf{U} \backslash \{0\} = \bigcup_{i=1}^{\infty} \{u \in \mathbf{U} |\ \langle u, \delta_i \rangle \neq 0\}.$$

Let $K_{i1} = K \cap [\delta_i < 0]$, $K_{i2} = K \cap [\delta_i > 0]$, $K_i = K_{i1} \cup K_{i2}$, $i = 1, 2, \ldots$. Hence $K \backslash \{0\} = \cup_{i=1}^{\infty} K_i$.

Let us first consider K_{i1}. Set $D_i = \{\delta_0, \delta_i\}$, $D_{i1} = \frac{1}{2}D + \frac{1}{2}D_i$, $x_{i1} = \frac{1}{2}x + \frac{1}{2}\delta_0$. Since $u(0) = 0$, we have $\langle u, \delta_0 \rangle = 0$, so that $K_{i1} \subset N(D_{i1}, x_{i1})$ and $\mu(K_{i1}) = \mu(N(D_{i1}, x_{i1}))$. Let $B(u_0, \epsilon) = \{u \in C(\mathbf{I}) |\ \|u - u_0\| < \epsilon\}$. Let $\tilde{x}_{i1} = \lambda x_{i1} + (1 - \lambda)\delta_0$, $\tilde{D}_{i1} = (\lambda D_{i1} \backslash \{x_{i1}\} + (1 - \lambda)\delta_i) \cup \{\tilde{x}_{i1}\}$ for some $\lambda \in (0, 1)$, and $\tilde{K}_{i1} = N(\tilde{D}_{i1}, \tilde{x}_{i1})$. We then have for any $u \in K_{i1}$,

$$\langle u, \tilde{x} \rangle < \langle u, \tilde{x}_{i1} \rangle \text{ for all } \tilde{x} \in \tilde{D}_{i1} \backslash \{\tilde{x}_{i1}\},$$

so that there exists some $\epsilon_u > 0$ such that $B(u, \epsilon_u) \cap \mathbf{U} \subseteq \tilde{K}_{i1}$. Letting $O_{i1} = \cup_{u \in K_{i1}} B(u, \epsilon_u)$, we obtain $K_{i1} \subseteq O_{i1}$ and $O_{i1} \cap \mathbf{U} \subseteq \tilde{K}_{i1}$. Furthermore, since μ is maximized at ρ, it follows that $\mu(\tilde{K}_{i1}) = \rho^{\tilde{D}_{i1}}(\tilde{x}_{i1})$ and $\mu(K_{i1}) = \rho^{D_{i1}}(x_{i1})$, hence, by continuity of ρ, that $\mu(\tilde{K}_{i1}) - \mu(K_{i1}) < \frac{\epsilon}{2^{i+1}}$ for λ sufficiently close to unity.

Likewise, we obtain for K_{i2} an open set O_{i2} and a set $\tilde{K}_{i2}$ such that $K_{i2} \subseteq O_{i2}$, $O_{i2} \cap \mathbf{U} \subseteq \tilde{K}_{i2}$, and $\mu(\tilde{K}_{i2}) - \mu(K_{i2}) < \frac{\epsilon}{2^{i+1}}$ for λ sufficiently close to unity. Letting $O_i = O_{i1} \cup O_{i2}$, $\tilde{K}_i = \tilde{K}_{i1} \cup \tilde{K}_{i2}$, we conclude that $K_i \subseteq O_i$, $O_i \cap \mathbf{U} \subseteq \tilde{K}_i$, where O_i is open in $C(\mathbf{I})$, and $\mu(\tilde{K}_i) - \mu(K_i) < \frac{\epsilon}{2^i}$ for λ sufficiently close to unity.

Repeating the above argument for all K_i, we will obtain

$$K \backslash \{0\} \subseteq \cup_{i=1}^{\infty} O_i,\ (\cup_{i=1}^{\infty} O_i) \cap \mathbf{U} \subseteq \cup_{i=1}^{\infty} \tilde{K}_i, \text{ and } \cup_{i=1}^{\infty} O_i \text{ is open in } C(\mathbf{I}).$$

Since S is compact in $C(\mathbf{I})$, so is $K \cap S$. As a consequence there exist a finite number of open sets, say $O_1, \ldots, O_m$, such that

$$K \cap S \subseteq \cup_{i=1}^{m} O_i,\ (\cup_{i=1}^{m} O_i) \cap \mathbf{U} \subseteq \cup_{i=1}^{m} \tilde{K}_i.$$

Let $O = \cup_{i=1}^{m} O_i$, $\tilde{K} = \cup_{i=1}^{m} \tilde{K}_i$; it is evident that $\tilde{K} \in \mathfrak{U}$, $K \cap S \subseteq O$, and $O \cap \mathbf{U} \subseteq \tilde{K}$. Note that $u \in \tilde{K}$ implies $\alpha u \in \tilde{K}$ for any $\alpha \geq 0$, so that $K \subseteq \tilde{K}$. Since μ is finitely additive it follows that $\mu(\tilde{K}) - \mu(K) < \epsilon$. This completes the proof. ■

6.5 Proof of Lemma 5.1 and Theorem 5.1

Let us begin by proving Lemma 5.1. For this fix t, r_0, and let

$$A_1 = \{u \in \mathbf{U} | r(u,t) < r_0\},\ A_2 = \{u \in \mathbf{U} | r(u,t) = r_0\}, \tag{6.9}$$

so that $A^t(r_0) = A_1 \cup A_2$. It suffices to show the measurability of A_1 and A_2.

Let us begin with A_2. Since in $\mathbf{U}$, any function is an increasing strictly concave transformation of another, it follows that A_2 is a singleton, say $A_2 = \{u_0\}$. Take $x_0 \in \mathbf{X}$ and consider the hyperplane

$$H = \{x \in \mathbf{X} | \langle u_0, x\rangle = \langle u_0, x_0\rangle\}$$

Since $\mathbf{X}$ associated with the Prokhorov metric is separable, so is H; let $\{x_1, x_2, \ldots\}$ be a countable dense subset of H. Define

$$D_n = \{x_{2n-1}, x_{2n}, y_n\}, n = 1, 2, \ldots,$$

where $y_n = \frac{1}{2}x_{2n-1} + \frac{1}{2}x_{2n}$. It is not hard to see that $u_0 = \cap_{n=1}^{\infty} N(D_n, y_n)$, hence that A_2 is measurable.

For the measurability of A_1, we note that $r_0 = 0$ implies $A_1 = \varnothing$, which is trivially measurable. Now suppose that $r_0 > 0$ and let

$$B_n = \{u \in \mathbf{U} | r(u,t) \leq r_0 - \frac{1}{n}\}, n = 1, 2, \ldots, \tag{6.10}$$

so that $B_1 \subset B_2 \subset B_3 \subset \cdots$ and $A_1 = \cup_{n=1}^{\infty} B_n$. It suffices therefore to show the measurability of each B_n. Again if $B_n = \varnothing$, it is trivially measurable. So assume $B_n \neq \varnothing$ and therefore there exists an $u^n \in \mathbf{U}$ such that $r(u^n, t) = r_0 - 1/n$. Take a nondegenerate $z_n \in \mathbf{X}$ such that $u_e^n(t) = u_e^n(z_n)$, where u_e^n is as defined in (2.1). Then applying Jensen's inequality we can conclude that $B_n = N(E_n, z_n)$, where $E_n = \{z_n, t\}$, and therefore B_n is measurable. This concludes the proof of Lemma 5.1.

We turn now to the proof of Theorem 5.1. We begin by proving $1 \Rightarrow 2$. Since μ_i is countably additive, it is continuous from below. Recall the definition of A_1, B_n from above; we have

$$F_i^t(r_0) = \mu_i(A^t(r_0)) = \mu_i(A_1) = \lim_{n\to\infty} \mu_i(B_n),$$

where the second equality follows from the regularity of μ_i. Since individual 1 is more risk averse in terms of RCR than individual 2 at t, it follows that $\mu_1(B_n) \leq \mu_2(B_n)$ for all n, hence that $F_1^t(r_0) \leq F_2^t(r_0)$.

We proceed to show $2 \Rightarrow 1$. Take any $D \in \mathfrak{D}_t$ and consider $N(D, t)$. Let

$$r_0 = \min\{r(u,t) | u \in N(D,t)\},$$

hence $N(D,t) = \{u \in \mathbf{U} | r(u,t) \geq r_0\}$. We have therefore

$$\rho_i^D(t) = \mu_i(N(D,t) = 1 - F_i^t(r_0),$$

which, along with individual 1 being more risk averse in terms of RUF than individual 2 at t, implies $\rho_1^D(t) \geq \rho_2^D(t)$.

A Appendix

A.1 Existence of a Regular RUF

In this subsection we shall take $\mathbf{U} = \mathbf{U}_1$ (for whose definition one may refer to Eq. (6.2)); the purpose is to show the existence of a regular RUF on $(\mathbf{U}, \mathfrak{U})$, and a similar argument holds also for the set $\mathbf{U}_2$. In the case of finite prizes, the existence of a regular RUF is demonstrated in Gul and Pesendorfer (ibid., Lemma 3) with the aid of the notion of volume. This notion however is not well defined in an infinite-dimensional setting, and so we have to look for a different method.

Observe that the notion of volume is a kind of measure, and Gul and Pesendorfer's lemma makes use of two properties of this measure: Consider the n-dimensional Euclidean space $\mathbb{R}^n$; then the volume of any open set of $\mathbb{R}^n$ is positive, and that of any set of dimension less than n is zero. So to look for a regular RUF on $(\mathbf{U}, \mathfrak{U})$ it suffices to look for a measure on it with the above two properties. The answer consists in the notion of a Radon Gaussian measure. We shall not present the precise definition of this notion, for which we refer to Bogachev (1998, Chapter 3), but just state two of its relevant properties. Recall that $\mathbf{U}$ is endowed with the supremum norm; we let $\mathfrak{B}(\mathbf{U})$ be the corresponding Borel σ-algebra on $\mathbf{U}$. Recall from Bogachev (ibid., Definition 3.6.2, p. 119) that a nondegenerate Radon Gaussian measure on $(\mathbf{U}, \mathfrak{B}(\mathbf{U}))$ is one that has $\mathbf{U}$ as its support; then according to its Problems 3.11.33 and 3.11.32 on page 154, we know that

PROPOSITION 2 There exists on $(\mathbf{U}, \mathfrak{B}(\mathbf{U}))$ a nondegenerate Radon Gaussian measure.

PROPOSITION 3 A Borel linear subspace in $\mathbf{U}$ has measure zero with respect to every nondegenerate Radon Gaussian measure on $\mathbf{U}$ precisely when it contains no continuously and densely embedded into $\mathbf{U}$ separable Hilbert space.

According to Proposition 2 it makes sense to let ϕ be a nondegenerate Radon Gaussian measure on $(\mathbf{U}, \mathfrak{B}(\mathbf{U}))$. By definition of nondegeneracy it follows that any open subset of $\mathbf{U}$ is of positive ϕ-measure. Now we show that any linear subspace of the form

$$H = \{u \in \mathbf{U} |\ \langle u, x\rangle = 0\},\ x \neq 0$$

is of zero ϕ-measure. To this end we note first that H is a Borel linear subspace, as $H = \cap_{n=1}^{\infty} H_n$, where

$$H_n = \{u \in \mathbf{U} |\ \langle u, x\rangle \in (-1/n, 1/n)\};$$

and secondly, according to Aliprantis and Border (op. cit., Lemma 5.55 and Corollary 5.81), that H is closed and not dense in $\mathbf{U}$. Then from Proposition 3 we infer that H is of zero ϕ-measure.

To summarize, we have demonstrated that every open subset of $\mathbf{U}$ has positive ϕ-measure and every linear subspace H has zero ϕ-measure. Using this fact and following through the argument of Gul and Pesendorfer (ibid., Lemma 3) (with ϕ as a substitute for their V) we can conclude the existence of a regular RUF on $(\mathbf{U}, \mathfrak{U})$.

A.2 Discontinuous Utility Functions

This subsection shows that when discontinuous utility functions are taken into account, the RUF that can be maximized by a given RCR may not be unique.

Specifically, consider the following family of discontinuous utility functions on [0, 1]

$$u_a(x) = -e^{ax} + \begin{cases} 1 & \text{if } x \geq \frac{1}{2} \\ 0 & \text{if } x < \frac{1}{2} \end{cases}, a \in [0, 1].$$

Let ν be some Borel measure on [0, 1] that admits a density. It is not hard to verify that for every $D \in \mathfrak{D}$ and every $x \in D$, the set

$$N_d(D, x) = \{a \in [0, 1] | \int_0^1 u_a(t) dx(t) \geq \int_0^1 u_a(t) dz(t) \text{ for all } z \in D\}$$

is a Borel set, where the subscript d is a shorthand for "discontinuous."It therefore makes sense to define a RCR ρ as follows:

$$\rho^D(x) = \nu(N_d(D, x)).$$

It then follows that there exists a RUF which is defined on the algebra generated by

$$\{N_d(D, x) | D \in \mathfrak{D}, x \in D\}$$

and which at the same time is maximized by ρ.

On the other hand, it can be checked without much difficulty that ρ is mixture continuous, monotone, linear and extreme. From Theorem 3.1, therefore, we may conclude that there exists a regular RUF on $(\mathbf{U}, \mathfrak{U})$ (whose definition is given in Section 2) which is maximized also by ρ. To summarize, we have constructed two distinct RUF's that are both maximized by the same RCR.

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