Expected Indirect Utility in an Ergodically Chaotic Overlapping Generations Model

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Abstract

The chaotic and ergodic equilibrium consumption profiles of a two period lived representative agent overlapping generations model are examined. Given a specific utility function, it is shown that for a typical equilibrium path expected indirect utility of consumption is less than the utility of expected equilibrium consumption. In turn, utility of expected consumption in equilibrium is less than utility at the steady state equilibrium. This result holds for a set of equilibrium maps of positive measure and suggests that stabilisation of the erratic system would bring about an improvement in welfare.

1 Motivation

In this paper the chaotic and ergodic equilibrium consumption profiles of a two period lived representative agent overlapping generations (OLG) model are examined. The utility function of the representative agent is assumed to be quasi-linear being quadratic in the non-linear additive component which pertains to consumption in the first period of life. Under these assumptions, for a subset of the set of parameters which determines the concavity of the utility function, it is known (Benhabib and Day)[1] that the equilibrium consumption path in the first period consumption of the lifecycle (and hence second period consumption) can exhibit topological chaos.

Topological chaos is characterised by a deterministic, non-linear, non-invertible, map which is hump-shaped in form. For such a map, if cycles of period three occur then, by a Theorem of Li and York [12], topological chaos exists for an

[1] The analysis of Benabib and Day [1] is based upon that of Gale [4] in which the latter author notes that in the OLG model the quadratic utility function can generate cycles of order two. Gale [4] comments that the existence of cycles of order two may be interpreted as a basic business cycle model. Benhabib and Day [1] take the work of [4] one step further by noting that not only are two-cycles present but so too are cycles of all orders as well as topological chaos.

uncountable scrambled set W. The trajectories are thus erratic; no trajectory can be predicted with any degree of precision at any meaningful horizon.

In the context of the OLG model of this paper, that the equilibrium path may be chaotic arises from the apposite choice of the endowment vector of the representative agent combined with the extent to which the quadratic utility of the first period consumption good is concave. Stated alternatively, topological chaos may occur to the extent that the intertemporal substitution effect dominates the income effect, or equivalently if the measure of risk aversion is large (Grandmont [5]) in the first period consumption good. In terms of bifurcation analysis, as the measure of concavity of the quadratic function increases, the equilibrium path bifurcates from a stable trajectory to cycles and to topological chaos interspersed with cycles of various orders.

The analysis of this paper, unlike that in [1], examines the existence of ergodic rather than topological chaos. Whilst both topological and ergodic chaos possess erratic and unpredictable trajectories, those of the latter have in addition a well-defined probability distribution invariant to the evolution of the dynamical system and the choice of starting values. Moreover, the probability has support of positive Lebesgue measure implying that ergodic chaos is an observable or physical phenomenon. For the OLG model examined here this means that the same distribution arises irrespective the precise equilibrium path or the point in which the equilibrium sequence commences.

By application of such principles it is demonstrated that

- i For a restricted subset of the set of parameters which determine the degree of intertemporal substitution or risk aversion and hence the concavity of the dynamical system, there exists a non-negligible set for which the equilibrium trajectories are ergodically chaotic.
- ii Along an equilibrium path the expected value of consumption in the first period of life is strictly less than the level of consumption at the steady state. This is due to the strict concavity of the equilibrium map.
- iii Given the linearity of the market clearing equation, expected consumption in the second period of life is strictly greater than consumption at the steady state.
- iv Since the utility function is strictly concave, (ii) and (iii) imply that expected indirect utility is strictly less than the value of indirect utility at the steady state.

Regarding point (iv), expected welfare as measured by expected indirect utility can be improved by stabilising the economic system to force convergence to the steady state under the proviso of course that the steady state remains invariant to any such stabilisation policy.

The discussion of this paper advances a theme in the economic literature on non-linear dynamics: the measurement of welfare in the presence of ergodic chaos and the sub-optimality of the latter with respect to a stable economic system. The paper is laid out as follows. In Section 2 the preliminaries are stated. In Section 3 the model and results are presented. Section 4 concludes.

2 Preliminaries

Let the state space $X \subset \mathrm{R}_+$ be a compact interval. At time $t \geq 0$ the state variable takes on values $x_t \in X$ and evolves in accordance with some continuous iterative rule h defined over X which determines the value of x_{t+1} given x_t; $x_{t+1} = h(x_t)$, written as $x_{t+1} = h^t(x)$ for some initial starting value $x \in X$. h is an interval map; $h : X \to X$ so all points in X remain in X under action of h. A trajectory or path $\{x_t\}_{t=0}^{\infty}$ is obtained by iterating $x_{t+1} = h(x_t)(= h^t(x))$ as $t \to \infty$ for some $x \in X$. A fixed point or steady state of $h : X \to X$ is a value $\bar{x} \in X$ such that $h(\bar{x}) = \bar{x}$.

It is sufficient for h to exhibit topological chaos if there exists some $x_1 \in X$ for which $x_2 = h(x_1)$, $x_3 = h^2(x_1)$ and $x_4 = h^3(x_1)$ such that $x_4 \leq x_1 < x_2 < x_3$, i.e. for $x_1 \in X$ h is periodic of order three [12]. Then there is an uncountable set $W \subset X$ containing no periodic point such that for every $x, y \in X$, $x \neq y$

i.

$$\lim_{t \to \infty} \sup \left| h^t(x) - h^t(y) \right| > 0$$

ii

$$\lim_{t \to \infty} inf \left| h^t(x) - h^t(y) \right| = 0$$

iii

For every $x \in W$ and every periodic point $y \in X$ one has

$$\lim_{t \to \infty} \sup \left| h^t(x) - h^t(y) \right| > 0$$

Stated alternatively, if h is topologically chaotic any two trajectories move apart from each other and close to each other under action of h. The trajectories are unpredictable or erratic. The set W is termed a scrambled set as the points of that set are scrambled or mixed as h evolves. It is apparent that such a dynamical system exhibits complex dynamics.

In order to motivate the use of ergodic rather than topological chaos, if $h : X \to X$ is topologically chaotic, as pointed out by Grandmont [5], the set W may be negligible in a measure theoretic sense thereby delimiting the significance of topological chaos as an observable or empirical phenomenon. The map h may therefore satisfy the conditions of topological chaos yet such erratic movements may never be observed. In such a case one talks of transient chaos for which the trajectory will settle down to a cycle of some order.

Instead if $h : X \to X$ exhibits ergodic chaos then the set W has has a non-negligible Lebesgue measure in X. The dynamics of h are observable or alternatively stated have a physical measure. In this sense ergodic chaos is an observable phenomenon (see [9] for a discussion).

More formally, if $h : X \to X$ is chaotic and ergodic then any two trajectories in X will converge to the same invariant density function $\varphi(x)$ with $\varphi(x) \geq 0 \quad \forall x \in X$ and $\int_X \varphi(x)dx = 1$. Furthermore, the support of φ has positive measure. This implies that the distribution has non-atomic mass at any point and fills up the support of φ. The dynamics of h are thus complex and observable. Given the preceding, for a typical trajectory $\{x_t\}_{t=0}^{\infty}$, x can be treated as a random variable. Furthermore the following properties are satisfied:

1. For any time invariant real-valued integrable function g defined over X, the time average of g is equal to the space average of g[2]

$$\lim_{t\to\infty} 1/T \sum_{k=1}^{T} g(h^k(x)) \to \int_X g(x)\varphi(x)dx, \quad a.e. \quad x \in X \tag{2.1}$$

where $E[g(x)] \equiv \int_X g(x)\varphi(x)dx$ is the expected value of $g(x)$. This is the Birkhoff – von Neumann mean converge theorem.

2. If, in (2.1), $g(x) = x$

$$\lim_{t\to\infty} 1/T \sum_{k=1}^{T} (h^k(x)) \to \int_X x\varphi(x)dx, \quad a.e. \quad x \in X \tag{2.2}$$

where $E[x] \equiv \int_X x\varphi(x)dx$ is the expected value of h, *i.e.* the time average is equal to the expected value). The expected values $E[g(x)]$ or $E[x]$ can be thought of as the long-run or limiting values of $g(h(x))$ and $h(x)$ respectively.

3. As a corollary of (2.1), h possesses the property of invariance; the time average of the state variable is invariant to the evolution of the dynamical system. Formally, for any time invariant integrable function g on X

$$E[g(x)] = E[g(h(x))] \text{ and in particular}$$

$$E[g(h(x))] = E[g(h^k(x))], \quad \forall k \geq 0 \tag{2.3}$$

(2.3) implies that the expected value of h is also invariant to the iteration of h

$$E[x] = E[h(x)] \text{ hence } E[h(x)] = E[h^k(x)] \quad \forall k \geq 0 \tag{2.4}$$

3 The Overlapping Generations Model and Chaotic Equilibria

The model presented here is the overlapping generations model examined by Gale [4] and subsequently by, *inter alia*, Benhabib and Day [1]. There is one consumption good per period and the representative agent lives for two periods

[2] The distribution may be derived from the empirical distribution of $h : X \to X$.

and maximises intertemporal utility over his lifecycle[3]. At time $t \geq 0$ the representative agent solves the maximisation problem

$$\max_{x_1^t, x_1^{t-1}} u(x_t^t, x_t^{t+1}) s.t. p_t x_t^t + p_{t+1} x_t^{t+1} \leq p_t \omega_1 + p_{t+1} \omega_2$$

where $x_t^\tau, \tau = t, t+1$ is the demand of the commodity in period $\tau = t, t+1$ of the lifecycle of the agent born at time $t, p_t > 0$ is the price of the commodity at time $t \geq 0$ and $\omega_\tau, \tau = 1, 2$ is the time independent endowment of the commodity. The utility function $u : R^2 \longrightarrow R$ is strictly concave, C^2 and satisfies standard boundary assumptions as well as being stationary or time independent. Equilibrium occurs at each point in time when the demand and supply of the coexisting young and old agents satisfies

$$x_{t-1}^t + x_t^t = \omega_1 + \omega_2 \tag{3.1}$$

Following an example in Gale [4] and Benhabib and Day [1], the utility function is quasi-linear being quadratic in the non-linear component of the function

$$u(x_t^t, x_t^{t+1}) = g(x_t^t) + x_t^{t+1} \tag{3.2}$$

where $g(x_t^t) = ax_t^t - \frac{1}{2}b(x_t^t)^2, a, b > 0$ and $0 \leq x_t^t \leq a/b$. It is noted that both g and u are strictly concave. For the sake of simplicity, it is assumed that the endowment vector is $(\omega_1, \omega_2) = (0, \omega_2)$ where $\omega_2 > a/b$. Solviing the maximisation problem for the specified utility function and utilising the equilibrium equation (3.2) yields the intertemporal equilibrium difference equation $x_{t+1}^{t+1} = h_{(a,b)}(x_t) = ax_t(1 - (b/a)x_t^t)$ parameterised by (a, b) written as

$$x_{t+1} = h_{(a,b)}(x_t) = ax_t(1 - (b/a)x_t) \tag{3.3}$$

where the superscript has been dropped and x_t is the equilibrium consumption in the first period of life of the agent born in time t. $h_{(a,b)}(x_t)$ thus generates an equilibrium trajectory $\{x_t\}_{t=0}^\infty$ in first period consumption.

In order to render the analysis illustrative of the concepts of interest, let $a = b = \mu$ such that the equation (3.3) defines the parameterised family of maps $h_\mu : X \to X$, $X = [0, 1]$ termed the logistic equation (see [2]) given as

$$h_\mu(x) = \mu x(1 - x) \tag{3.4}$$

[3] In the terminology of Gale [4], the OLG model is Classical, i.e. the representative agent consumes more in the first period of life than the second period of life. This is to be contrasted with the Samuelson model in which more is consumed in the second period of life. The two models produce differing sequences of equilibria. In the case of the former, equilibrium paths are defined moving forward in time and for the latter equilibrium paths are defined moving backwards in time. Given that the equilibrium map in both cases is non-invertible then in the Classical case the backwards dynamics are not well-defined whereas in the Samuelson case the forward dynamics are not well-defined.

where the parameter can take on values $\mu \in (0,4]$ and governs the position, gradient and degree of concavity of h_μ (see [2])[4].

The fixed points of h_μ are the trivial fixed point or autarky (no trade and consumption of the endowment vector) $\bar{x}_\mu = 0 = \omega_1$, $\forall \mu \in (0,4]$ and the non-trivial fixed point or steady state $\bar{x}_\mu = (\mu-1)/\mu$, $\forall \mu \in (1,4]$ where $\bar{x}_\mu > 0 = \omega_1$ so trade takes place.

In the case of the trivial fixed point $h\prime_\mu(0) = \mu$ so $|h\prime_\mu(0)| \leq 1 \quad \forall \mu \in (0,1]$ and $|h\prime_\mu(0)| > 1 \quad \forall \mu \in (1,4]$ hence the origin $\bar{x}_\mu = 0$ is stable for any $\mu \in (0,1]$ and is a repelling fixed point and not converged upon for any $\mu \in (1,4]$ and all initial values $x \in X\backslash\{0\}$.

At the steady state $h_\mu(\bar{x}_\mu) = (\mu-1)/\mu$. Such that the steady state exists it is necessary that $\mu > 1$ otherwise the unique fixed point is the origin which is stable for $\mu \in (0,1]$ hence ergodic chaos cannot exist. Since $h\prime_\mu((\mu-1)/\mu) = 2-\mu$ then $|h\prime_\mu((\mu-1)/\mu)| < 1$ for $\mu \in (1,3)$ and $|h\prime_\mu((\mu-1)/\mu)| > 1$ for $\mu \in (3,4]$. Therefore, if (3.4) is chaotic and in particular ergodically chaotic it is necessary (but not sufficient) that $\mu \in (3,4]$.

As a matter of example, let $\mu = 4$. It is well known (see [2], [10] and [13]) that for this particular parameterisation of h_μ, the trajectory of the dynamical system generates the density function $\varphi(x) = \left[\pi\sqrt{x(1-x)}\right]^{-1}$ for $x \in (0,1)$. Calculation of the expected value shows that $E[x] = \lim_{\substack{s\to 0\\ t\to 1}} \int_s^t x\varphi(x)dx = 0.5$ and the steady state $\bar{x} = 0.75$. Hence the expected value of first period equilibrium consumption is less than the value of first period equilibrium consumption at the steady state

$$E[x] < \bar{x} \tag{3.5}$$

The inequality of (3.5) is suggestive in that one is led to question the implications of this on the value of utility of the representative agent along an equilibrium path. Before tackling the relationship suggested by (3.5), a more general question needs to be answered as to whether (i) the inequality in (3.5) holds for parameter values other than $\mu = 4$ and (ii) if so, whether the set of such parameters of $\mu \in (3,4]$ is non-negligible in a measure theoretic sense?

To answer the first question assume that there exists a subset of $\mu \in (3,4]$, denoted Λ, for which the first period equilibrium consumption determined by (3.4) is ergodically chaotic. Denote as $\varphi_\mu(x)$ the probability distribution of the trajectory of h_μ; $\{x_t\}_{t=0}^\infty$, which is parameterised by $\mu \in \Lambda$ and denote the expected value of the equilibrium sequence $\{x_t\}_{t=0}^\infty$ as $E_\mu[x] = \int_X x\varphi_\mu(x)dx$ which is again parameterised by $\mu \in \Lambda$. Under the assumption that Λ is non-empty, the following argument demonstrates that (3.5) holds for every $\mu \in \Lambda$.

Let $\mu \in \Lambda$. Then h_μ is ergodically chaotic so φ_μ exists and has support of positive measure.

[4] If $\mu > 4$ then h_μ is not an interval map as $h_\mu : X \to Y$ where X is a strict subset of Y. In such a case the dynamics of h_μ escape X. On the other hand, $X = [0,1]$ for all $\mu \in (0,4]$ and $h_\mu([0,1]) = [0, \frac{1}{4}\mu]$, hence $h_\mu([0,1]) \subseteq [0,1]$ for all $\mu \in (0,4]$ so there always exists at least one fixed point of h_μ.

Consider that $var_\mu[x] \in (0,1)$. To see this suppose first that $var_\mu[x] = 0$ for some $\mu \in \Lambda$. Then the iterates h_μ are at a fixed point or periodic point of finite order. It follows that h_μ is not chaotic and so $\mu \notin \Lambda$; a contradiction of the assumption.

Suppose now that $var_\mu[x] \geq 1$. Taking expectations of both sides of $h_\mu(x) = \mu x(1-x)$ yields $E_\mu[h_\mu(x)] = E\mu x(1-x)] = \mu E_\mu[x] - \mu E_\mu[x_2]$. By the assumed ergodicity of (3.4) and given that the probability distribution is invariant to h_μ then, by (2.4), $E_\mu[h_\mu(x)] = E_\mu[x]$. It follows that $E_\mu[x] = \mu E_\mu[x] - \mu E_\mu[x^2]$. Rearranging this last expression gives $E_\mu[x^2] = E_\mu[x](\mu-1)/\mu \quad \equiv E_\mu[x]\bar{x}_\mu$.

Since $var_\mu[x] \geq 1$ then $E_\mu[x^2] - (E_\mu[x])^2 \geq 1$ which when rearranged and given the preceding, $E_\mu[x](\mu-1)/\mu \geq 1 + (E_\mu[x])^2$ or $E_\mu[x](\mu-1) \geq \mu + \mu(E_\mu[x])^2$ which implies that $E_\mu[x] \geq \mu/(\mu-1)$. But since $\mu \in \Lambda \subset (3,4)$ then $E_\mu[x] \geq \mu(\mu-1) > 1$. However, $h_\mu : X \to X$, $X = [0,1]$ so $E_\mu[x] \in (0,1)$ (since $h_\mu^{-1}(0) = \{0,1\}$ then $E_\mu[x] = 0$ and $E_\mu[x] = 1$ are not possible). A contradiction. It follows that $var_\mu[x] \in (0,1)$.

Now, by $var_\mu[x] \in (0,1)$ then $var_\mu[x] = E_\mu[x^2] - E_\mu[x])^2 > 0$ therefore $E_\mu[x^2] > E_\mu[x])^2 > 0$. Using again the fact that $E_\mu[x^2] = E_\mu[x](\mu-1)/\mu \equiv E_\mu[x]\bar{x}_\mu$ as well as $E_\mu[x^2] > E_\mu[x])^2 > 0$ one has that $E_\mu[x^2] = E_\mu[x]\bar{x}_\mu > (E_\mu[x])^2$ hence $E_\mu[x] < \bar{x}_\mu$. The following Lemma holds[5].

> **Lemma 3.1.** Consider the map $h_\mu : X \to X$, where $X = [0,1]$ defined by $h_\mu(x) = \mu x(1-x)$. Let $\mu \in \Lambda$ be the set of parameters such that h_μ is ergodically chaotic. Then
>
> $$E_\mu[x] < \bar{x}_\mu, \quad \forall \mu \in \Lambda \tag{3.6}$$
>
> In establishing Lemma 3.1 it is assumed that the set Λ is non-empty; an assumption validated by setting $\mu = 4$ in the above discussion. In order to show that the set Λ has a non-negligible measure a result of Jakobson [10] is evoked.

Consider a quadratic family of maps[6] where a member of that family is a smooth function $h_\mu(x) : x \to \mu x(1-x)$ where $\mu \in (0,4]$ and $h_\mu(x) = \mu h(x)$ with $h(x)$ sufficiently close to $x(1-x)$ in $C^3([0,1],[0,1])$. Jakobson [10] considered the set of parameter values μ for which the family of quadratic maps h_μ is ergodically chaotic. The following theorem is stated.

> **Theorem 3.1. Jakobson Theorem B [10].** Let $h_\mu(x)$ belong to the quadratic family of maps. Then there is a positive measure of Λ

[5] This argument could have been demonstrated by utilizing the fact that h_μ is a strictly concave function over X. Since h_μ is strictly concave, an application of Jensen's inequality for strictly concave functions implies that $h_\mu(\int_X h_\mu(x)\varphi_\mu(x)dx) > (\int_X h_\mu(h_\mu(x))\varphi_\mu(x)dx)$. As $\mu \in \Lambda$, ergodicity and invariance of h_μ implies that $E_\mu[h_\mu(x)] = E_\mu[x]$ hence $(\int_X h_\mu(h_\mu(x))\varphi_\mu(x)dx) = (\int_X h_\mu(x)\varphi_\mu(x)dx) = (\int_X x\varphi_\mu(x)dx) = E_\mu[x]$. By the last inequality one has that $h_\mu E_\mu[x]) > E_\mu[x]$. Given that h_μ is unimodal and strictly concave it follows that $h_\mu E_\mu[x]) > E_\mu[x] \Leftrightarrow E_\mu[x] < \bar{x}_\mu$. A similar argument for convex functions appears in [6].

[6] The quadratic family of maps is not to be confused with the quadratic utility function.

such that for each $\mu \in \Lambda$, h_μ admits a stable and invariant density function.

Jakobson's Theorem can be employed to establish that for the family of quadratic maps, for any $\varepsilon > 0$ there exists a $\rho > 0$ such that[7]:

$$m\{\mu \in \Lambda : \mu_0 \geq \mu \geq \mu_0 - \rho\} > \rho(1-\varepsilon)$$

where m denotes the one-sided Lebesgue measure of $\mu \in \Lambda$. Hence $\Lambda \subset (3, 4]$ has a non-negligible Lebesgue measure.

By Theorem 3.1, it follows that Lemma 3.1 holds for a set Λ of positive measure and the inequality of (3.6) therefore holds for a non-negligible set of parameterisations of the dynamical system. As a consequence of this result it is meaningful to discuss the implications of the relationship between the expected value of consumption and the non-trivial fixed point as if $\mu \in (0, 4]$ is chosen at random then with positive probability h_μ is both ergodic and chaotic and (3.6) holds.

To be precise, of interest is the manner in which ergodic chaos impacts on the welfare of the representative economic agent where welfare is measured in terms of expected lifetime consumption along a typical equilibrium path.

As a matter of abbreviation, denote first and second period consumption respectively as x_1 and x_2. By Theorem 3.1, for each $\mu \in \Lambda$ one has $E_\mu[x_1] < \bar{x}_{1,\mu}$ where $\bar{x}_{1,\mu}$ is the steady state first period consumption obtained from (3.5); $\bar{x}_{1,\mu} = (\mu - 1)/\mu$[8]. Since the equilibrium equation (3.1) holds for each $t \geq 0$ then $E_\mu[x_1] + E_\mu[x_2] = \omega_1 + \omega_2 = \bar{x}_{1,\mu} + \bar{x}_{2,\mu}$ hence $E_\mu[x_2] > \bar{x}_{2,\mu}$. Therefore by the Birkhoff – von Neumann theorem on average, less is consumed in the first period of life than would be consumed at the steady state and more is consumed in the second period of life than would be consumed at the steady state.

The implications of the foregoing can be framed in terms of contrasting the measure of welfare along the chaotic equilibrium path with that of the steady state equilibrium. Taking expectations of the utility function

$$E_\mu[u(x_1, x_2)] = \mu E_\mu[x_1] - \frac{1}{2}\mu E_\mu[(x_1)^2] + E_\mu[x_2] \tag{3.7}$$

Given that equilibrium consumption allocations satisfy $E_\mu[x_1] + E_\mu[x_2] = \omega_1 + \omega_2 = 0 + \omega_2$, (3.7) can be written as

$$E_\mu[u(x_1, x_2)] = \mu E_\mu[x_1] - \frac{1}{2}\mu E_\mu[(x_1)^2] + \omega_2 - E_\mu[x_1] \tag{3.8}$$

It is required to show that $E_\mu[u(x_1, x_2)] < u(\bar{x}_{1,\mu}, \bar{x}_{2,\mu})$. Assume the contrary; that $E_\mu[u(x_1, x_2)] \geq u(\bar{x}_{1,\mu}, \bar{x}_{2,\mu})$ stated explicitly as

$$\mu E_\mu[x_1] - \frac{1}{2}\mu E_\mu[(x_1)^2] - E_\mu[x_1] \geq \mu \bar{x}_{1,\mu} - \frac{1}{2}\mu(\bar{x}_{1,\mu})^2 - \bar{x}_{1,\mu}$$

[7] Remark XIII/5 pg. 87 [10].

[8] Similarly, $\bar{x}_{2,\mu}$ is the second period equilibrium consumption at the fixed point obtained from the value of $\bar{x}_{1,\mu}$ and expression (3.1).

Rearranging the preceding one has that

$$\begin{aligned}\left[(\bar{x}_{1,\mu})^2 - E_\mu[(x_1)^2]\right] &\geq 2\left((\mu-1)/\mu\right)[\bar{x}_{1,\mu} - E_\mu[x_1]] \Leftrightarrow (\bar{x}_{1,\mu})^2 \\ -E_\mu[(x_1)^2] &\geq 2(\bar{x}_{1,\mu})^2 - 2\bar{x}_{1,\mu}E_\mu[x_1]\end{aligned}$$

Since $E_\mu[(x_1)^2] = \bar{x}_{1,\mu}E_\mu[x_1]$ the last expression is then equivalent to

$$(\bar{x}_{1,\mu})^2 - \bar{x}_{1,\mu}E_\mu[x_1] \geq 2(\bar{x}_{1,\mu})^2 - 2\bar{x}_{1,\mu}E_\mu[x_1]$$

which implies $E_\mu[x_1] \geq \bar{x}_{1,\mu}$, a contradiction of Lemma 3.1. The following is therefore obtained.

$$E_\mu[u(x_1, x_2)] < u(\bar{x}_{1,\mu}, \bar{x}_{2,\mu}) \tag{3.9}$$

(3.9) states that for each $\mu \in \Lambda$, the expected level of utility is less than the utility at the steady state[9]. On average therefore, a representative agent faces a lower level of indirect utility in equilibrium which is ergodically chaotic than that which could be obtained at the steady state.

Given the preceding, one could argue that a representative agent is not cognisant of the fact that the on average welfare could be improved; the benchmark against which welfare is measured, the steady state, is never observed as the dynamical system never converges upon the steady state equilibrium. As such, if a social planner were to be introduced, such a normative judgement could be made and there is a positive case for the stabilisation of the dynamical system.

4 Conclusion

This paper brings together two threads which remain underdeveloped in the economic literature on non-linear dynamics.

(i) The existence of ergodic chaos and the statistical properties of a deterministic equilibrium system.

(ii) The manner in which ergodic chaos impacts on welfare the representative agent.

To date, little has been written in relation to ii given i, the analysis of Kennedy *et al.* [11] being, to this author's knowledge, one such analysis.

Concerning the first argument, it has been demonstrated that there exists a non-negligible set of parameters which govern the degree of concavity of the

[9] It is interesting to note that (3.9) can be calculated as the average of the utility of all future representative agents. To see this note that at some $t \geq 0$ the representative agent's indirect utility in equilibrium is $E_\mu[u(x_t^t, x_t^{t+1})] = E_\mu[g(x_t^t)] + E_\mu[x_t^{t+1}]$. By the equilibrium equation; $x_{t+1}^{t+1} + x_t^{t+1} = \omega_2$, and by the fact that ergodicity implies $E_\mu[x_t^t] = E_\mu[x_{t+1}^{t+1}]$ then expected indirect utility can be written as $E_\mu[u(x_t^t, x_t^{t+1})] = E_\mu[u(x_t^t, x_t^t)] = E_\mu[g(x_t^t)] + \omega_2 - E_\mu[x_t^t]$. By utilising expressions (2.1) and (2.2) this last expression can be written as $E_\mu[u(x_t^t, x_t^{t+1})] = \lim_{T\to+\infty} T^{-1}\sum_{k=1}^T g(h^k(x_t)) + \omega_2 - \lim_{T\to+\infty} T^{-1}\sum_{k=1}^T h^k(x_t)$. Hence expected utility can be calculated as the average of the indirect utility along an equilibrium path for any representative agent. The last expression is consistent with the fact that the equilibrium sequence is obtained under the assumption of perfect foresight.

dynamical system for which the probability measure derived from the equilibrium trajectory is invariant to the evolution of the dynamical system. As such, the moments of the distribution obtained from the equilibrium trajectory then exist for non-negligible sets of parameters. It is thus meaningful to enquire into the statistical properties of the ergodically chaotic equilibrium trajectories

It is stressed however that the analysis is based on the restriction in the quadratic utility function that $\{(a,b) \in \mathrm{R}^2_{++} : a = b\}$from which $a = b = \mu$ was made. The set of parameters of the dynamical system $\mu \in \Lambda$ for which h_μ is ergodic and chaotic has thus positive Lebesgue measure. However, in the space of parameters which define the quadratic component of the utility function this set has zero measure. A more thorough analysis would consider whether the results extend to the set $\{(a,b) \in \mathrm{R}^2_{++} : a \neq b\}$. This can be examined by application of a theorem contained in Grandmont [5], pg. 63.

Concerning the second argument, the strict concavity of the equilibrium map was used in conjunction with the equilibrium equation to establish a relation between $E_\mu[x_i]$, $i = 1, 2$ and the fixed points. The question was then put forward as to the manner in which expected indirect utility along an equilibrium trajectory behaved with respect to indirect utility at the steady state. It was shown that on average, if the equilibrium system is ergodically chaotic, welfare is less than welfare at the steady state. This result is in part an artefact of the choice of utility function being quadratic from which the resultant equilibrium map is the logistic equation which possesses well-established results such as Theorem 3.1.

It is thus a point of interest whether the results of this paper hold for the same OLG model but with a differing utility function. Similarly, one could ask whether similar results hold for differing models and differing dynamical systems as well as whether such results exist for a non-negligible set of parameters in the sense of Theorem 3.1[10].

In terms of extending the results of Section 3, it is noted that inequality (3.9) holds for topologically chaotic and periodic equilibrium trajectories hence for a larger set of parameters than Λ.

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[10] It would be interesting to examine how expected utility behaves with respect to the fixed point of an equilibrium system if that system is not concave but say convex. In the context of a one-dimensional tatonnement process the map of which is strictly convex, Huang [6] finds that the inequality of (3.6) is reversed. Given the optimisation problem of a representative agent, on the basis of the enquiry herein one is led to ask how expected utility behaves with respect to the stable equilibrium if (3.6) is reversed.

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