The Interrelationship between the Non-Existence of Stationary Sunspot Equilibria and Stationary Equilibria

1. Introduction

In a model of exchange, uncertainty may influence decision making in various ways. Often, as is assumed, uncertainty appears as a determinant of the fundamental structure of the economy, affecting preferences, endowments, demographic structure, production and so on. In such a case, equilibrium prices and allocations reflect uncertainty as rational agents modify behaviour conditional upon the realisation of an uncertain event. Uncertainty is *intrinsic* to the economic system.

Uncertainty is *extrinsic* to the economic system if randomness *does not* affect any of the economic fundamentals but has instead a direct and unique bearing only on prices. That is, agent's rational decision making reflects the common belief that prices are determined by some stochastic process which does not affect any of the aforementioned fundamentals of the economy. The resultant equilibrium is then a rational expectations equilibrium under extrinsic uncertainty. Expectations are deemed to be self-fulfilling, requiring nothing more than a common outlook on the price process. The equilibria are termed sunspot equilibria (Cass and Shell []).

Various models of extrinsic uncertainty or sunspot have been analysed, from a model in which there is restricted market participation [5], to dynamic [19] and stationary models within a framework of an overlapping generations (OLG) structure [2] [7] [20]. In particular, in a stationary overlapping generations model in which labour is transformed into the production of a single commodity, Azariardis and Guesnerie [2] show that stationary sunspot equilibria[1] (SSE) exist and are dense[2] in the space of probabilities if and only if cyclical equilibrium of order 2 exist.

Guesnerie [10] extends the results of [2] by examining and showing the existence of sunspots for the case of multiple commodities. In [10], the (multi-vector) form of the excess demand *does not*, however, incorporate historic prices (and is hence not applicable to an OLG model); the model in [10] is a one step forward looking model which excludes the presence of lagged prices. The exclusion of the possibility of lagged prices restricts the generality of the model and delimits its applicability.

In [20], Spear shows that for a two period stationary pure exchange overlapping generations (OLG) economy with one commodity and either separable or non-separable utility functions, SSE exist for a dense set of probabilities. The sufficient condition for the existence of SSE is that the elasticity of saving of the first period consumption good

[1] Equilibria which are sunspot and are time invariant.
[2] Sunspot equilibria exist for a non-negligible Lebesgue measure of probabilities.

evaluated at the monetary steady state is less than -1/2 when the Markovian probability structure is degenerate and equivalent to the occurrence of 2-cycles. It is thus implicit that 2-cycle equilibria are sufficient for the existence of SSE in a pure exchange environment.

Putting aside the sunspot framework, in a similar but distinct vein, within the framework of a stationary pure exchange OLG model in which there are multiple commodities each period and in which the endowment structure is stochastic, (uncertainty is intrinsic and not extrinsic as the fundamentals of the economy depend upon the occurrence of an uncertain event) [3], Spear [21] discusses the existence of stochastic rational expectations equilibrium. It is shown that within the space of parameters defined as the space of utility functions and endowments, stationary stochastic equilibria do not generically[4] exist. However, if the utility functions are time separable then non-stochastic stationary steady state equilibria do exist whereas if the utility function is time non-separable then it is conjectured that stationary non-stochastic equilibrium will typically not exist. Furthermore, if there is one commodity each period then non-stochastic as well as stochastic equilibrium can, under certain conditions, exist.

Typically, in order to show that non-existence is generic one usually applies Thom's transversality Theorem (see for instance [8]). This theorem states that the set of functions for which the image mapping of the system intersects the desired solution set transversally[5] is dense in the space of possible functions which may give rise to the system of equations. Given that transverse intersection is a generic property, in the case in which there are more variables than unknowns, transverse intersection occurs when there is no intersection[6]. As Spear ([21] pg. 263) points out however, a direct application of Thom's transversality theorem to the framework of stationary OLG with a stochastic endowment structure is not possible as the restriction in the space of possible functions giving rise to excess demand equations imposed by requiring that agents be expected utility maximisers are such that they preclude the direct application of this theorem. Stated alternatively, given the additive structure of the expected utility functions as well as the requirement that equilibrium allocations must differ across states of nature[7], Thom's theorem cannot be readily employed. Instead, the manner in which Spear shows that stochastic rational expectations equilibria do not exist is by using a multijet version of Thom's Theorem, termed the multijet transversality theorem (reported in [8]) whereby the first order conditions and market clearing conditions are used to show that for an open and dense set of *utility functions* and endowment vectors equilibrium fails to exist as there are simply too many equilibrium equations given the number of equilibrium price vectors; the system of equilibrium equations is overdetermined.

[3] The stochastic process is therefore not sunspot. The model of [21] is however similar to the multi-commodity sunspot model as examined in this paper.
[4] A property is said to be generic in a space if it holds for an open and dense subset of that space.
[5] In the case of a system of excess demand functions, the desired solution set is the 0 vector.
[6] Transverse intersection is a generic property.
[7] If this were not the case then the equilibrium would be non-stochastic and hence the equilibrium system would be degenerate.

Several points are in order. Whilst Spear's analysis is illuminating[8] , the application of the results of [21] to the framework of extrinsic uncertainty does require qualification. Secondly, as demonstrated in this paper, Spear's non-existence result when applied to the case of extrinsic uncertainty can be shown to holds in a simple manner without the requirement of the somewhat difficult machinery of multijet spaces. That is to say, for a given utility function and almost every endowment vector, SSE can be shown to not exist by means of a simple transversality and dimension counting argument. Thirdly, Spear [21] finds that if the utility function is separable in first and second period arguments, then a steady state equilibrium exists, yet no stationary stochastic equilibrium exists. If instead the utility function is not time separable, stochastic equilibria do not exist yet it is not clear whether or not the steady state equilibrium generically exists. Given the stochastic structure of the endowment process, Spear [21] conjectures that in the case of non-separable utility functions, the steady state equilibrium does not exist. The question arises as to whether this result applies to the framework of extrinsic uncertainty. For the sunspot model, it is shown that this result need not necessarily hold.

In sum, this paper is motivated by the following observations.

i. The application of the non-existence result to the extrinsic uncertainty framework requires examination and qualification.

ii. The demonstration of the generic non-existence of SSE can be achieved by means of a simple transversality argument. The advantage of this approach is that it is both intuitive and holds for the entire endowment space. The disadvantage is that the result is not dense in the space of utility functions.

iii. Non-stochastic steady state exist arbitrarily near to a space of economies for which stationary sunspot equilibria do not exist. Furthermore, the existence of stationary steady state equilibria does not require that the utility functions be time separable.

This paper is laid out as follows. In Section 2 the stationary overlapping generations model is presented and discussed. The budget manifold and equilibrium manifold are defined and shown to be submanifolds of the price-income space for a fixed level of total resources. In Section 3 a result of Golubitsky and Guillemin is evoked which states that if the combined dimensions of two manifolds or spaces is less than the ambient space in which the two manifolds reside, then these two manifolds can intersect transversally only if they do not intersect at all. This result is then applied to show that the transversal intersection of the budget manifold and the equilibrium manifold can only occur if the intersection is an empty one. This implies that there is no set of prices and incomes which can simultaneously satisfy both the budget constraint and the system of equilibrium equations. Consequently, stationary sunspot equilibria do not generically exist. In Section 4, the question is posed that if stochastic equilibria do not exist, do non-stochastic equilibria exist? It is shown that if the

[8] See Nagata [17] for an application of the multijet transversality theorem in the context of a pure exchange framework to show the generic existence of equilibria in the space of utility functions.

Markovian probabilities are degenerate and the system of equilibrium equations collapses to one which is equivalent to those of a pure exchange economy then for almost every endowment vector, equilibrium exist. It is then shown that for almost every endowment vector a steady state equilibrium exists. Hence, coincidental with the generic non-existence of stationary sunspot equilibria one has that non-sunspot (non-stochastic) equilibria exist. A brief review of transversality is contained in the Appendix.

2.1 The Stationary Sunspot Model

The model is a stationary[9] stochastic overlapping generations model in which uncertainty enters the model by means of each agent's commonly held belief that a randomly determined signal influences prices. Such a process is termed sunspot[10]. As such, randomness does not have any bearing on the fundamentals of the system, i.e. demographic structure, preferences, endowment structure[11], but instead operates solely by means of a commonly held belief that prices affect consumption plans and other decisions[12]. This commonly held belief is expressed by means of a public forecast function $\sigma(\cdot)$ which assigns, in a one-to-one manner, a price vector p_i to a commonly observed random signal s_i; $p_i = \sigma(s_i)$. Each signal, the occurrence of which is termed an event, belongs to a finite signal set. For ease of exposition and without loss of generality it is assumed that there are two signals. In each period a representative agent is born and lives for two periods[13]. At any point in time there are then 2 agents coexisting one of whom is young the other of whom is old. Uncertainty unfolds by means of a stationary Markov process where the probability of an agent observing a sequence of events over a life-cycle is given by $\pi_{ss'}$ where $\pi_{ss'}$ is the probability of state *s'* being realised conditional on the realisation of signal *s*. Given that there are two signals then $\{s,s'\} \in \{1,2\} \times \{1,2\}$. The sum of all the probabilities of all the events over any given life-cycle is equal to one. It is assumed that every conditional probability is bound between $(0,1)$ and is hence non-degenerate.

There are $L > 1$ commodities each period. An agent born in state *s*, has first period demand vector $x_1^s \in \mathbb{R}_{++}^L$, $x_1^s = (x_1^s, x_2^s, ..., x_L^s) \in \mathbb{R}_{++}^L$, for $s \in \{1,2\}$, and has second period contingent demand vector $x_2^{ss'} \in \mathbb{R}_{++}^L$ $x_2^{ss'} = (x_{2,1}^{ss'}, x_{2,2}^{ss'}, ..., x_{2,L}^{ss'}) \in \mathbb{R}_{++}^L$, for $s \in \{1,2\}$. Over an entire life-cycle each agent has vectors of demands; $x^s = (x_1^s, x_2^{ss'}, x_2^{s's'}) \in \mathbb{R}_{++}^L \times \mathbb{R}_{++}^{2L}$, $s \neq s'$ for a fixed first period $s \in \{1,2\}$. Since there are 2 diverse agent types (one born in each state; an

[9] Time invariant.
[10] See *inter alia* [2] and [6]
[11] This type of uncertainty is said to be extrinsic. Uncertainty is intrinsic if randomness influences the real variables of the economic system.
[12] Expectations are rational.
[13] The analysis follows through if the agent lives for any finite number of periods. The assumption of a two period lived agent is made for the sake of simplicity.

agent is henceforth identified by the state in which he is born), then the space of contingent commodity vectors is $(x^1, x^2) \in \mathbb{R}^{2L}_{++} \times \mathbb{R}^{4L}_{++}$.

Given the above definition of the price forecast function, to each event a price vector is assigned. If $\pi_{ss'}$ is the probability of the of state *s'* being realised conditional on the realisation of signal *s* then the price assigned is $p_{s'}$ for any $s \in \{1,2\}$ where $p_s = (p_{1,s}, ..., p_{L,s}) \in \mathbb{R}^L_{++}$, $s = 1,2$. Each agent is endowed with a set of non-stochastic endowment vectors of the *L* commodity goods over the 2 periods of life; $\omega = (\omega_1, \omega_2) \in \Omega \equiv \mathbb{R}^{2L}_{++}$. As uncertainty is extrinsic, the utility function $u : \mathbb{R}^{2L}_{++} \to \mathbb{R}$ is state independent where $u^{-1}(a)$ is bounded from below for all $a \in \mathbb{R}_{++}$. Furthermore, $cl\{x' \in \mathbb{R}^{2L}_{++} : u(x') \geq u(x)\} \subset \mathbb{R}^{2L}_{++}$. It is also assumed that *u* is smooth, i.e. $Du \in C^{\infty}(\mathbb{R}^{LI}_{++}, \mathbb{R}^{LI}_{++})$, i.e. *u* has positive derivatives. *u* is strictly concave and has negative definite Hessian, i.e. $y^T \cdot D^2u(x) \cdot y < 0$ for all $x \in \mathbb{R}^{2L}_{++}$ and $y \in \mathbb{R}^{2L}$, $y \neq 0$.

The maximisation problem for the agent born into states 1 and 2 are respectively :

<u>Problem 1 – Agent 1:</u>

$\max_{x_1^1, x_2^{1j}} \pi_{11} u(x_1^1, x_2^{11}) + \pi_{12} u(x_1^1, x_2^{12})$ s.t. $p_1 \cdot x_1^1 + p_m x_m^1 \leq p_1 \cdot \omega_1$ and $p_j \cdot x_2^{1j} + \leq p_j \cdot \omega_2 + p_m x_m^1$, for $j = 1,2$ (2.1)

<u>Problem 2 – Agent 2:</u>

$\max_{x_1^2, x_2^{2j}} \pi_{21} u(x_1^2, x_2^{21}) + \pi_{22} u(x_1^2, x_2^{22})$ s.t. $p_2 \cdot x_1^2 + p_m x_m^2 \leq p_2 \cdot \omega_1$ and $p_j \cdot x_2^{2j} + \leq p_j \cdot \omega_2 + p_m x_m^2$, for $j = 1,2$ (2.2)

p_m is the price of money. The demand for money of an agent born into state *j* is $x_m^j = \left(p^j / p_m\right) \cdot (\omega_j^1 - x_1^j)$, $j = 1,2$. $p_s \in \mathbb{R}^L_{++}$, $s = 1,2$ is the price of the commodity in state *s*. Denote $p = (p_1, p_2) \in S \equiv \mathbb{R}^{2L}_{++}$ as the price space. Let price of money be normalised to 1; $p_m \equiv 1$. Define $I = [0,1]$. Then $\pi = \text{int}(\pi_{12}, \pi_{21}) \in \text{int}(I^2)$ defines the set of all permissible non-degenerate Markovian probabilities. As a matter of definition a pair $(\omega, \pi) \in \Omega \times \text{int}(I^2)$ is termed a <u>*sunspot economy*</u>.

2.2. The Price-Income Space and the Budget Manifold

By the elimination of the money demand, the constraints in maximisation Problems 1 and 2 can be written respectively as (2.3) – (2.4) and (2.5) – (2.6):

$$p_1 x_1^1 + p_1 x_2^{11} = p_1 \omega_1 + p_1 \omega_2 \tag{2.3}$$

$$p_1 x_1^1 + p_2 x_2^{12} = p_1 \omega_1 + p_2 \omega_2 \tag{2.4}$$

$$p_2 x_1^2 + p_1 x_2^{21} = p_2 \omega_1 + p_1 \omega_2 \tag{2.5}$$

$$p_2 x_1^2 + p_2 x_2^{22} = p_2 \omega_1 + p_2 \omega_2 \tag{2.6}$$

By the previous set of constraints income is stochastic and dependent upon the state into which the agent is born, the probability of the realisation of the event as well as the price vector. Define $p_i \cdot \omega_j = w_{ij}$, $\{i, j\} \in \{1,2\} \times \{1,2\}$. Then write stochastic incomes as[14]:

$$p_1 \cdot \omega_1 + p_1 \cdot \omega_2 = w_{11} + w_{12} \tag{2.3}$$

$$p_1 \cdot \omega_1 + p_2 \cdot \omega_2 = w_{11} + w_{22} \tag{2.4}$$

$$p_2 \cdot \omega_1 + p_1 \cdot \omega_2 = w_{21} + w_{12} \tag{2.5}$$

$$p_2 \cdot \omega_1 + p_2 \cdot \omega_2 = w_{21} + w_{22} \tag{2.6}$$

It is assumed that total resources are fixed at some value $r \in \mathbb{R}_{++}^L$ where $\omega_1 + \omega_2 = r$. Given $r > 0$ the space of endowments is restricted by $\Omega(r) \equiv \{\omega \in \Omega : \ \omega_1 + \omega_2 = r\}$. The total value of resources is then $(p_1, p_2) \cdot r$ for some $\omega \in \Omega(r)$. Denote $B = S \times \mathbb{R}_{++}^4$ as the *price-income* space where $(p, w) = (p_1, p_2, w_{11}, w_{12}, w_{21}, w_{22}) \in B$ is a generic element. The dimension of this space is $\dim B = 2L + 4$. For a specific endowment vector $\omega \in \Omega(r)$, the following holds:

$$\begin{aligned}(p_1, p_2) \cdot (\omega_1, \omega_2) &= p_1 \cdot (\omega_1 + \omega_2) + p_2 \cdot (\omega_1 + \omega_2) \\ &= p_1 \cdot r + p_2 \cdot r \\ &= w_{11} + w_{12} + w_{21} + w_{22} \end{aligned} \tag{2.7}$$

[14] It is noted that (2.3) and (2.4) (also (2.5) and (2.6)) are linearly independent iff $p_1 \neq p_2$. Indeed, as is noted subsequently, a sunspot equilibrium can exist iff $p_1 \neq p_2$. Conversely, if $p_1 = p_2$ then any resultant equilibrium cannot be sunspot and is characterised by linear dependence of the constraints.

For some $\omega \in \Omega(r)$, $(p_1, p_2)\cdot(\omega_1, \omega_2) = w_{11} + w_{12} + w_{21} + w_{22}$ is a polynomial of order 1 in (p, w). By the requirement that total resources be fixed, the price income-space B is thereby restricted to the subset of $(p, w) \in B$ such that (2.7) holds. Denote this subset of B as $B(r)$ where it is noted that $B(r)$ is an affine subspace or linear manifold, and hence submanifold, of B with $\dim B(r) = 2L + 3$. The subspace $B(r)$ is termed the *restricted price-income space*.

The *budget manifold* is the set $A(\omega)$ of price-income pairs such that (2.3) – (2.6) are satisfied. $A(\omega)$ is an linear subspace of B where $\dim A(\omega) = 2L$. Furthermore, for $(p, w) \in A(\omega)$ one has that (2.3) and (2.6) hold. By summing these two constraints one has that $p_1 \cdot \omega_1 + p_1 \cdot \omega_2 + p_2 \cdot \omega_1 + p_2 \cdot \omega_2 = w_{11} + w_{21} + w_{12} + w_{22}$. The same applies to the summation of the constraints (2.4) and (2.5). Thus if $(p, w) \in A(\omega)$ then $(p, w) \in B(r)$ so $A(\omega)$ is a $2L$ dimensional linear subspace of $B(r)$, itself a $2L + 3$ dimension subspace of B.

2.3. The Equilibrium Manifold

The solutions to maximisation Problems 1 and 2 yield the respective vectors of demands $f^j : S \times \Omega \times I \to \mathbb{R}^3_{++}$, $j = 1, 2$ where $f^1 = (f_1^1, f_2^{11}, f_2^{12})$ and $f^2 = (f_1^2, f_2^{21}, f_2^{22})$ given by the following two $3L$ vectors:

$$f^1(p_1, p_2, p_1\omega_1 + p_1\omega_2, p_1\omega_1 + p_2\omega_2, \pi_{12})$$

$$f^2(p_1, p_2, p_2\omega_1 + p_1\omega_2, p_2\omega_1 + p_2\omega_2, \pi_{21})$$

where f^j, $j = 1, 2$ is smooth, bounded from below and satisfies desirability[15]. It is assumed that $\pi \in \text{int}(I^2)$ is fixed and does not appear parametrically in the demand functions. Given the above definition of stochastic income, for a pair $(p, \omega) \in S \times \Omega(r)$, each agent's vector of demand functions is the map $f^j : S \times \mathbb{R}^2_{++} \to \mathbb{R}^{3L}_{++}$ defined by:

$$f^1(p_1, p_2, w_{11} + w_{12}, w_{11} + w_{22})$$

$$f^2(p_1, p_2, w_{21} + w_{12}, w_{21} + w_{22})$$

[15] The function f^j satisfies desirability if, for any sequence $(p_{i(q)}, p_{j(q)}, w_i^{i(q)}, w_i^{j(q)}, \pi_{ji}^{(q)}) \in S \times \mathbb{R}^2_{++} \times I$ where $p_i\omega_{i,1} + p_i\omega_{i,2} = w_i^i$ and $p_j\omega_{j,1} + p_j\omega_{j,2} = w_i^j$ has a convergent sequence which converges to $(p_{i(0)}, p_{j(0)}, w_i^{i(0)}, w_i^{j(0)}, \pi_{ji}^{(0)}) \in S \times \mathbb{R}^2_{++} \times I$ where at least one of the elements of $(p_{i(0)}, p_{j(0)})$ are equal to zero, then $\lim_{q\to\infty} \sup \left\| f^j(p_{i(q)}, p_{j(q)}, w_i^{i(q)}, w_i^{j(q)}, \pi_{ji}^{(q)}) \right\| = +\infty$.

Equilibrium in the sunspot model occurs when demand is equal to supply across all goods and across all states of nature as well as the money markets being in a state of equilibrium. Equilibrium is a *sunspot equilibrium* if $p_1 \neq p_2$ (i.e. prices reflect the extrinsically uncertain event) and the equilibrium allocations differ across both states of nature for at least one agent for at least one good. This condition is given by the simultaneous satisfying of the system of equations (2.8) – (2.13) subject to the requirement that $p_1 \neq p_2$ and $(f_1^1, f_2^{11}) \neq (f_1^1, f_2^{12}) \neq (f_1^2, f_2^{21}) \neq (f_1^2, f_2^{22})$ [16].

$$f_1^1(p_1, p_2, w_{11}+w_{12}, w_{11}+w_{22}) + f_2^{11}(p_1, p_2, w_{11}+w_{12}, w_{11}+w_{22}) - \omega_1 - \omega_2 = 0 \tag{2.8}$$

$$f_1^1(p_1, p_2, w_{11}+w_{12}, w_{11}+w_{22}) + f_2^{21}(p_1, p_2, w_{21}+w_{12}, w_{21}+w_{22}) - \omega_1 - \omega_2 = 0 \tag{2.9}$$

$$f_1^2(p_1, p_2, w_{21}+w_{12}, w_{21}+w_{22}) + f_2^{12}(p_1, p_2, w_{11}+w_{12}, w_{11}+w_{22}) - \omega_1 - \omega_2 = 0 \tag{2.10}$$

$$f_1^2(p_1, p_2, w_{21}+w_{12}, w_{21}+w_{22}) + f_2^{22}(p_1, p_2, w_{21}+w_{12}, w_{21}+w_{22}) - \omega_1 - \omega_2 = 0 \tag{2.11}$$

$$f_m^1(p_1, p_2, w_{21}+w_{12}, w_{21}+w_{22}) = -p_1 \cdot (\omega_1 - f_1^1(p_1, p_2, w_{11}+w_{12}, w_{11}+w_{22})) = \bar{m} \tag{2.12}$$

$$f_m^2(p_1, p_2, w_{21}+w_{12}, w_{21}+w_{22}) = -p_2 \cdot (\omega_1 - f_1^2(p_1, p_2, w_{21}+w_{12}, w_{21}+w_{22})) = \bar{m} \tag{2.13}$$

where $\bar{m}$ is the pre-existing money supply. (2.8) – (2.11) are the goods market equations and (2.12) and (2.13) are the money market equations. Henceforth the role of the money markets is overlooked. This is justified as if equilibrium does not obtain in the goods market then there cannot be equilibrium in the money market as this latter market is redundant. It is with no loss of generality that this is assumed as the following analysis could be replicated for the system of equations defined by (2.8) – (2.13).

Let the system of demands (2.8) – (2.11) be divided into two subsystems given by the map $z_j : B(r) \to \mathbb{R}^{2L}$ $j = 1, 2$ defined as:

$$z_1(p_1, p_2, w_{11}, w_{12}, w_{21}, w_{22}) = \begin{cases} f_1^1(p_1, p_2, w_{11}+w_{12}, w_{11}+w_{22}) + f_2^{11}(p_1, p_2, w_{11}+w_{12}, w_{11}+w_{22}) - r \\ f_1^2(p_1, p_2, w_{21}+w_{12}, w_{21}+w_{22}) + f_2^{22}(p_1, p_2, w_{21}+w_{12}, w_{21}+w_{22}) - r \end{cases} \tag{2.14}$$

[16] It is noted by the equations (2.8) – (2.11), in equilibrium $f_2^{11} = f_2^{21}$ and $f_2^{12} = f_2^{22}$. Hence $f_2^{11} \neq f_2^{12}$ iff $f_2^{21} \neq f_2^{22}$ iff $f_1^1 \neq f_1^2$ which implies that the equilibrium is sunspot iff $(f_1^1, f_2^{11}) \neq (f_1^1, f_2^{12}) \neq (f_1^2, f_2^{21}) \neq (f_1^2, f_2^{22})$ which can be readily checked by the first order conditions to be the case iff $p_1 \neq p_2$.

$$z_2(p_1,p_2,w_{11},w_{12},w_{21},w_{22}) = \begin{cases} f_1^1(p_1,p_2,w_{11}+w_{12},w_{11}+w_{22}) + f_2^{21}(p_1,p_2,w_{11}+w_{12},w_{11}+w_{22}) - r \\ f_1^2(p_1,p_2,w_{21}+w_{12},w_{21}+w_{22}) + f_2^{12}(p_1,p_2,w_{21}+w_{12},w_{21}+w_{22}) - r \end{cases} \tag{2.15}$$

It is noted that z_j is a smooth map. The *equilibrium manifold*, denoted $B_j(\omega)$, is the subset of $B(r)$ such that $z_j(p,w)=0$. $B_j(\omega)$, $j=1,2$ is a smooth submanifold of $B(r)$ of dimension 2. To see that $B_j(\omega)$ is a smooth submanifold, the Regular Value Theorem is utilised[17]. By the fact that z_j is a smooth map, then $z_j^{-1}(0)$ is a manifold of dimension $\dim B(r) - \dim \mathbb{R}^{2L} = 2L+3-2L = 3$ of $B(r)$. Now, consider $j=1$ and note that in equilibrium $f_1^1 + f_2^{11} = \omega_1 + \omega_2$ and $f_1^2 + f_2^{22} = \omega_1 + \omega_2$. Multiply the first L equations by p_1 and the second L equations by p_2; $p_1 \cdot f_1^1 + p_1 \cdot f_2^{11} = p_1 \cdot \omega_1 + p_1 \cdot \omega_2$, $p_2 \cdot f_1^2 + p_2 \cdot f_2^{22} = p_2 \cdot \omega_1 + p_2 \cdot \omega_2$. Summing the latter two expressions yields:

$$\begin{aligned} p_1 \cdot f_1^1 + p_1 \cdot f_2^{11} + p_2 \cdot f_1^2 + p_2 \cdot f_2^{22} &= p_1 \cdot \omega_1 + p_1 \cdot \omega_2 + p_2 \cdot \omega_1 + p_2 \cdot \omega_2 \\ &= w_{11} + w_{12} + w_{21} + w_{22} \end{aligned} \tag{2.16}$$

By Walras Law, if $(p,w) \in z_1^{-1}(0)$ is a price-income pair which clears the market then (2.16) is also satisfied. Therefore the dimension of the equilibrium manifold is reduced by 1. It follows that $B_1(\omega)$ is a 2-dimensional submanifold of $B(r)$. Analogous reasoning shows that $B_2(\omega)$ is also a smooth submanifold of $B(r)$ of dimension 2.

3.1 The Interaction of the Budget Manifold and the Equilibrium Manifold

Equilibrium is defined as the subset of $(p,w) \in B(r)$ for which the budgets are maintained and the excess demand functions attain their zeroes. Stated formally, $(p,w) \in B(r)$ is an equilibrium if and only if

$$(p,w) \in A(\omega) \cap B_1(\omega) \cap B_2(\omega) \tag{3.1}$$

[17] The Regular Value Theorem (Guillemin and Pollack [11] or Milnor [15]) states that if $f : X \to Y$ is a smooth map between smooth manifolds and $y \in Y$ is a regular value of f then the preimage $f^{-1}(y)$ is a smooth submanifold of X with $\dim f^{-1}(y) = \dim X - \dim Y$.

Equilibrium therefore requires a non-empty intersection between the budget manifold and the equilibrium manifolds. Specifically, $(p,w) \in B(r)$ is a sunspot equilibrium if and only if $p_1 \neq p_2$ and equilibrium allocations differ across the states of nature for at least one agent for at least one good (see footnote 16 on pg. 8). Sunspot equilibria therefore require the non-empty intersection of $A(\omega)$ with $B_1(\omega)$ and $A(\omega)$ with $B_2(\omega)$. It is now claimed that generically, i.e. for almost all $\omega \in \Omega(r)$ and $r > 0$, *sunspot equilibria do not exist*. In order to show this a simple transversality and counting argument is employed. The following defines transversal intersection.

Definition 1. (Definition 4.1 Golubitsky and Guillemin pg. 50 [8]). Let *X* and *Z* be smooth manifolds and $f : X \to Z$ a smooth mapping. Let *Y* be a smooth submanifold of *Z* and *x* a point of *X*. Then *f* intersects *Y* transversally at *x* if either

i. $f(x) \notin Y$, or

ii. $f(x) \in Y$ and $T_{f(x)}W + (df)_x(T_x X) = T_{f(x)}Y$ (where $T_x X$ is the tangent space to *X* at *x* and likewise for $T_{f(x)}W$ and $T_{f(x)}Y$)

In light of Definition 1 consider as an example two manifolds *X* and *Y* both of which belong to a space *Z* which may or may not be a manifold itself of some other space. *X* and *Y* are said to be transversal to each other if they intersect non-tangentially. Symbolically one writes $X \pitchfork Y$. For instance, suppose that *X* and *Y* are two curves in $\mathbb{R}^2$ which cut each other (e.g. let *X* and *Y* be the two axes of $\mathbb{R}^2$). It is easy to see that *transversal intersection is a generic property*; if *X* and *Y* cut each other non-tangentially at a point then a small perturbation does not disturb the property of non-tangential intersection. In such a case, the tangent spaces of *X* and *Y* span the ambient space (ii of Definition 1). More generally, *X* and *Y* will typically be transversal to each other; in the case in which *X* and *Y* do intersect each other tangentially then a small perturbation of either *X* or *Y* or both suffices to restore the property of transversality. Therefore, transversal intersection is a generic property (Guillemin and Pollack pp. 67 [11]). Furthermore, the intersection of two manifolds is itself a manifold with the dimension of the manifold satisfying[18] $\text{codim}(X \cap Y) = \text{codim}(X) + \text{codim}(Y)$ (in the previous example the intersection of the axes *X* and *Y* is a single point which is itself a 0-dimensional manifold).

However, as pointed out in Guillemin and Pollack [11] (also Hirsch pg. 67 [12] and Golubitsky and Guillemin pg. 51 [8]) the transversal intersection of *X* and *Y* depends not only on the manner in which the two surfaces or manifolds interact but also on the ambient space in which they reside. Returning to the previous example, suppose that *X* is the horizontal axes

[18] The codimension of *X* in *Z* is $\text{codim}(X) = \dim Z - \dim X$.

and Y the vertical axes in $\mathbb{R}^2$. Then X and Y are both manifolds of dimension 1[19] with $\dim X + \dim Y = \dim Z$; the sum of the dimensions of the two manifolds is equal to the dimension of the ambient space. Moreover, X and Y are transverse to each other and a movement in one or the other (or both) lines will generally not disrupt the property of transversal intersection.

Suppose now that X and Y are still two lines (e.g. assume that they are two axes as above), but now belong to $\mathbb{R}^3$. It is noted that the dimensions of these curves are such that $\dim X + \dim Y = 2 < 3 = \dim Z$. Then the intersection of X and Y is not a generic property; assume that X and Y intersect. Then a slight movement in either X or Y suffices to pull the two lines off each other in a manner such that the intersection between the two curves is empty. It follows that transversal intersection, being a generic property (Definition 1.i), implies that the intersection of X and Y does not typically occur (see Appendix for a brief discussion and illustration of this point) [20]. Hence, transversal intersection implies non-intersection; $X \pitchfork Y \Rightarrow X \cap Y = \varnothing$.[21] Formally:

<u>Proposition 1 (Golubitsky and Guillemin, Proposition 4.2, pg. 51 [8]).</u> Let X and Z be smooth manifolds and $Y \subset Z$ a submanifold. Suppose that $\dim X + \dim Y < \dim Z$ (i.e. $\dim(X) < \operatorname{codim}(Y)$). Let $f : X \to Z$ be smooth and suppose that $f \pitchfork Y$. Then $f(X) \cap Y = \varnothing$.

It suffices to let f be the identity map in Proposition 1 in order to state the following:

$$\dim(X) < \operatorname{codim}(Y) \Rightarrow X \cap Y = \varnothing \tag{3.2}$$

(3.2) may occur if the dimension of either X or Y is not large enough relative to the ambient space Z. Turning now to the budget and equilibrium manifolds, application of the foregoing will show that equilibrium does not generically exist. Let $\omega \in \Omega(r)$. Then $(p, w) \in B(r)$ is an equilibrium if and only if (3.1) is satisfied. Now, $\operatorname{codim}(A(\omega)) = \dim B(r) - \dim(A(\omega)) = 3$ and $\dim(B_j(\omega)) = 2$, $j = 1, 2$. Hence:

$$\dim(B_j(\omega)) < \operatorname{codim}(A(\omega)), \ j = 1, 2 \tag{3.3}$$

(3.3) therefore implies that $B_j(\omega) \cap A(\omega) = \varnothing$, $j = 1, 2$.

[19] X and Y are 1-dimensional manifolds as the real line is a 1-dimensional manifold.
[20] If X and Y do not intersect each other then X and Y cannot be tangential to each other. Given the definition of transversality, it follows that if X and Y do not intersect each other then they are transversal to each other, hence a transversal intersection may be an empty intersection.
[21] Guillemin and Pollack pg. 30 – 32. [11].

It is concluded that there does not exist (p, w) common to both $B_j(\omega)$ and $A(\omega)$ from which it follows that the equilibrium set is empty. Since $\omega \in \Omega(r)$ and $r > 0$ were arbitrarily chosen, (3.3) generically holds.

If it is the case that there is indeed some $(p, w) \in A(\omega) \cap B_1(\omega) \cap B_2(\omega)$ with $p_1 \neq p_2$ then by reasoning analogous to that underlying the motivating discussion of Proposition 1, a slight perturbation of $\omega \in \Omega(r)$ will suffice to pull the manifolds away from each other thereby restoring the condition (3.3) (see Figure 3 in the appendix for such an illustration). Finally, it is noted that since equilibria do not exist in the goods market then equilibrium in the money market, whether it exists or not, is redundant as there is no need for the use of money. Theorem 1 summarises.

<u>Theorem 1.</u> In the case of $L > 1$ commodities, sunspot equilibria do not generically exist in the stationary OLG model.

<u>4. Proximity of Certainty Equilibria to the Non-Existence of Sunspot Equilibria</u>

Theorem 1 was derived under the assumption that $\pi \in \text{int}(I^2)$ is fixed and equilibrium satisfy the sunspot hypothesis which requires that $p_1 \neq p_2$. The question arises as to whether non-stochastic equilibria can exist alongside the non-existence of sunspot equilibria. Non-sunspot or certainty economies may occur when the Markov probability structure is degenerate, e.g. when $(\pi_{12}, \pi_{21}) = (1,1)$. Given the imposition of this probability vector, the maximisation problems become:

<u>Problem 1(i): – Agent 1:</u>

$$\max_{x_1^1, x_2^{12}} u(x_1^1, x_2^{12}) \text{ s.t. } p_1 \cdot x_1^1 + p_2 \cdot x_2^{12} \leq p_1 \cdot \omega_1 + p_2 \cdot \omega_2 \tag{4.1}$$

<u>Problem 2(i) – Agent 2:</u>

$$\max_{x_1^1, x_2^{21}} u(x_1^2, x_2^{21}) \text{ s.t. } p_2 \cdot x_1^2 + p_1 \cdot x_2^{21} \leq p_2 \cdot \omega_1 + p_1 \cdot \omega_2 \tag{4.2}$$

The resultant vector of demand functions for agents 1 and 2 are respectively:

$$f^1 = (f_1^1(p_1, p_2, p_1 \cdot \omega_1 + p_2 \cdot \omega_2)), f_2^{12}(p_1, p_2, p_1 \cdot \omega_1 + p_2 \cdot \omega_2)) \in \mathbb{R}_{++}^{2L}$$

$$f^2 = (f_1^2(p_1, p_2, p_2 \cdot \omega_1 + p_1 \cdot \omega_2), f_2^{21}(p_1, p_2, p_2 \cdot \omega_1 + p_1 \cdot \omega_2)) \in \mathbb{R}^{2L}_{++}$$

The system of excess demand is the mapping $(z_1, z_2): S \times \mathbb{R}^{2L}_{++} \to \mathbb{R}^{2L}$ where $S \equiv \mathbb{R}^{2L}_{++}$ is the non-normalized price space and is given by the functions:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} (f_1^1(p_1, p_2, p_1 \cdot \omega_1 + p_2 \cdot \omega_2) + f_2^{21}(p_1, p_2, p_2 \cdot \omega_1 + p_1 \cdot \omega_2) - \omega_1 - \omega_2 \\ f_1^2(p_1, p_2, p_2 \cdot \omega_1 + p_1 \cdot \omega_2) + f_2^{12}(p_1, p_2, p_1 \cdot \omega_1 + p_2 \cdot \omega_2) - \omega_1 - \omega_2 \end{pmatrix} \quad (4.3)$$

The question arises as to whether there exists a solution to (4.3). In [22] it is shown that stationary OLG economies have an equivalence to pure exchange economies. It is easy to see that the maximisation problems (4.1) and (4.2) have the form of a pure exchange model in which there are two consumers and $2L$ goods with the restriction that both consumers have the same utility and that the consumption and endowment vectors are permuted, i.e. agent one has the consumption and endowment vector (x_1^1, x_2^{12}) and (ω_1, ω_2) and agent two has the consumption vector (x_2^{21}, x_1^2) and (ω_2, ω_1) [22]. The structure of the economy, being somewhat particular, is termed a cyclical pure exchange economy [22]. The economy however remains a pure exchange economy to which one can apply the battery of well-established pure exchange economy results. The following two propositions concerning pure exchange economies are evoked.

Proposition 1 (Propositions 2.7.1 and 2.7.2 Balasko [3]). The set of regular economies[23] R in Ω is open and dense with full measure.

Proposition 2 (Corollary 4.6.4, Balasko [3]). The number of equilibria of a regular economy $\omega \in R$ is odd.

[22] By translating the OLG model into a pure exchange model, agent 1's first period consumption vector becomes the first L goods and the second period consumption vector becomes the second L goods whereas agent 2's second period consumption vector becomes the first L goods and the first period consumption vector becomes the first L goods. Thus the stationary OLG model in which there are L goods over two time periods and two agents is equivalent to a pure exchange economy in which there is one time period and $2L$ goods and two agents.

[23] A regular economy is a vector $\omega \in \Omega$ for which the Jacobian of the set of excess demand functions does not lose rank. Proposition 1 is an application of Sard and Brown's Theorem. See Milnor [] for a concise explanation of these two theorems.

Proposition 1 states that generically, in the pure exchange model, under regular assumptions (concavity of utility functions, boundedness of demand functions and desirability), regular economies are typical or occur with measure 1[24]. Proposition 2 states that for any regular economy the number of equilibrium price vectors are odd. Since 1 is an odd number, for almost every $\omega \in \Omega$ there exists at least one equilibrium price vector. It thereby follows that there is an odd number of equilibria to the system of equations defined by (4.3). Hence for $(\pi_{12}, \pi_{21}) = (1,1)$ equilibrium generically exist; for any $\omega \in \Omega(r)$ the intersection of $A(\omega)$ with $B(\omega)$[25] is non-empty and by the odd number of equilibria for any regular economy $B(\omega) \cap A(\omega)$ is a 0-dimensional manifold[26] comprised of an odd number of points. Such equilibria are termed <u>*stationary equilibria*</u>[27]. However, despite the fact that stationary equilibrium generically exist for any $\omega \in \Omega(r)$, $r > 0$ given $(\pi_{12}, \pi_{21}) = (1,1)$ (and are hence structurally stable being robust to perturbations in the parameter space of endowments), for any $\pi \in \text{int}(I^2)$ near to $(1,1)$ <u>*sunspot equilibria*</u> do not exist.

It is thus concluded that arbitrarily close to the robust existence of equilibria, sunspot equilibria do not exist. This conclusion is however reached given the requirement that equilibrium prices have the property that $p_1 \neq p_2$. It is now shown that for every $\omega \in \Omega(r)$, $r > 0$ and every $\pi \in \text{int}(I^2)$ there exists an equilibrium price vector of the form $(p_1^*, p_1^*) \in \mathbb{R}_{++}^{2L}$.

Consider the case of $(\pi_{12}, \pi_{21}) = (1,1)$. If (p_1, p_2) is a solution to (4.3) then so is the vector (p_2, p_1) (see [22] for a demonstration of this point). Such equilibria are termed <u>*cyclical*</u>[28]. Thus for every equilibrium vector of the form (p_1, p_2) $p_1 \neq p_2$, there exists the permuted counterpart (p_2, p_1). By the fact that there exists an odd number of equilibria (Proposition 2), there must exist a "tie breaking" equilibrium price vector which satisfies $(p_1, p_2) = (p_2, p_1)$ or $p_1 = p_2$ thereby yielding the equilibrium vector (p_1^*, p_1^*). Specifically, by the fact that equilibria are odd in number for any $\omega \in \Omega$, there must always exists an equilibrium of the form (p_1^*, p_1^*) (where of course the precise values of the equilibrium price vector will depend on the value of ω).

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[24] An economy which is not regular is termed singular and gives rise to a critical equilibrium. A critical equilibrium may be characterised by a continuum of price vectors. However such cases arise with measure 0.

[25] $B(\omega)$ is the equilibrium manifold derived from the system of equations (4.3) and $A(\omega)$ is the budget manifold derived from the constraints in Problems (4.1) and (4.2).

[26] The fact that the intersection of the two manifolds is a 0-dimensional manifold follows from the fact that generically the number of equilibria are odd hence finite in number and locally isolated.

[27] The term stationary equilibria is due to the fact that these equilibria are fixed points of the stationary OLG model.

[28] Specifically in this context such equilibria are termed cyclical pure exchange equilibria. By the equivalence to stationary OLG models such equilibria occur if the OLG model exhibits predictable fluctuations of cycles of order 2.

Yet, such a price vector (p_1^*, p_1^*) is an equilibrium for *any* $\pi \in \text{int}(I^2)$ not just the degenerate case of $(\pi_{12}, \pi_{21}) = (1,1)$ which gives rise to the cyclical pure exchange economy. To see this note that if $p_1 = p_2$ then the two constraints in each of the maximisation Problems 1 and 2 above are equivalent. Hence each agent maximises the stochastic utility function subject to the same constraints twice which is equivalent to maximising the certainty utility function subject to a single constraint. The two maximisation problems are reported:

<u>Problem 1(i): – Agent 1:</u>

$$\max_{x_1^1, x_2^{12}} u(x_1^1, x_2^{12}) \text{ s.t. } p_1 \cdot x_1^1 + p_1 \cdot x_2^{12} \le p_1 \cdot \omega_1 + p_1 \cdot \omega_2 \tag{4.4}$$

<u>Problem 2(i) – Agent 2:</u>

$$\max_{x_1^1, x_2^{21}} u(x_1^2, x_2^{21}) \text{ s.t. } p_1 \cdot x_1^2 + p_1 \cdot x_2^{21} \le p_1 \cdot \omega_1 + p_1 \cdot \omega_2 \tag{4.5}$$

The resultant demand functions from (4.4) and (4.5) are identical to those derived from (4.1) and (4.2) apart from the fact that the vector p_2 in the latter two problems has been substituted with p_1. Hence a solution to (p_1^*, p_1^*) to (4.3) is also a solution to the system of excess demand functions derived from (4.4) and (4.5) which implies that (p_1^*, p_1^*) is an equilibrium (4.3) the (p_1^*, p_1^*) is an equilibrium for any $\pi \in \text{int}(I^2)$. Moreover, as previously noted such an equilibrium always exists.

Theorem 1 must consequently be read with some caution as it does not state that generically no equilibrium exist in the sunspot model as for any $\pi \in \text{int}(I^2)$ and almost all $\omega \in \Omega$, there necessarily exist an equilibrium price vector of the form (p_1^*, p_1^*). Instead, there do not exist sunspot equilibrium in the sunspot model hence there do not exist equilibrium with (p_1^{**}, p_2^{**}) where $p_1^{**} \ne p_2^{**}$. One thus has that $(p_1^*, p_1^*, w_1^*, w_2^*) \in A(\omega) \cap B_1(\omega) \cap B_2(\omega)$ with $w_1^* = w_2^*$. This result can be contrasted against Spear [21] in which a stationary steady state exists only if the utility function is separable across time-periods.

The assumption of extrinsic uncertainty therefore imposes a stationarity in the endowment vectors which implies that if $p_1 = p_2$ the constraints of the maximisation problem are linearly dependent and only one constraint binds The stochastic utility function then reduces to a certain utility function for which the resultant system of excess demand functions necessarily have an equilibrium. That this outcome obtains is particular to the structure of uncertainty.

The following observations conclude.

i. Stationary sunspot equilibria do generically not exist. What is more, non-existence was demonstrated by means of a straight forward transversality argument.

ii. Given the cyclical pure exchange economy or certain economy derived from the extrinsic stochastic framework, it has been demonstrated that for almost every $\omega \in \Omega$ stationary equilibria exist[29]. This implies that in the case of the underlying certainty economy the intersection of the budget manifold and the equilibrium manifold is non-empty, i.e. one has that there exists at least one pair $(p, w) \in A(\omega) \cap B(\omega)$. Given the fact that there exists an open and dense set of $\pi \in \text{int}(I^2)$ in the vicinity of $(\pi_{12}, \pi_{21}) = (1,1)$, one can state that in the vicinity of every certainty equilibrium, an uncountable number of sunspot equilibria do not exist.

iii. Every cyclical pure exchange or certainty economy has an stationary equilibrium price vector of the form (p_1^*, p_1^*) which is also equilibrium of the extrinsic uncertainty economy. Thus given any $\pi \in \text{int}(I^2)$, (p_1^*, p_1^*) is an equilibrium but does not embed the extrinsically determined events (i.e. p_1^* is the same across states of nature $s = 1, 2$) and hence is not a sunspot equilibrium.

iv. Point iii of this conclusion was obtained without the need to make any assumption concerning the additive separability of the utility functions.

v. Computation shows that the discussion of this paper readily extends to the case in which the state space is comprised of any finite number of signals as well as representative agents who live for more than 2 periods.

Suggestions for Future Research:

Sunspot equilibria do not exist as there are simply too many equations given the number of unknowns. A possible solution to this impasse is to perhaps create a parameter space say K such that the system of excess demand functions (2.14) and (2.15) has the form of $z : S \times \Omega \times K \to \mathbb{R}^{4L}$ where $\dim S = 2L$, $\dim \Omega = 2L$ and $\dim \mathbb{R}^{4L} = 4L$. The equilibrium manifold is then the set of $(p, \omega, k) \in S \times \Omega \times K$ such that $z(p, \omega, k) = 0$ and is the set $E = z^{-1}(0)$. Such that z generates a well-defined manifold then the space K needs to be of dimension $4L$ embedded in the space $S \times \Omega \times K$ ($\dim(S \times \Omega \times K) - \dim(\mathbb{R}^{4L}) = 4L$). Under regularity assumptions E is then a smooth manifold of dimension $4L$ which belongs to the space $S \times \Omega \times K$. Define the projection map from E to the space $\Omega \times K$ by the map $\chi : E \to \Omega \times K$.

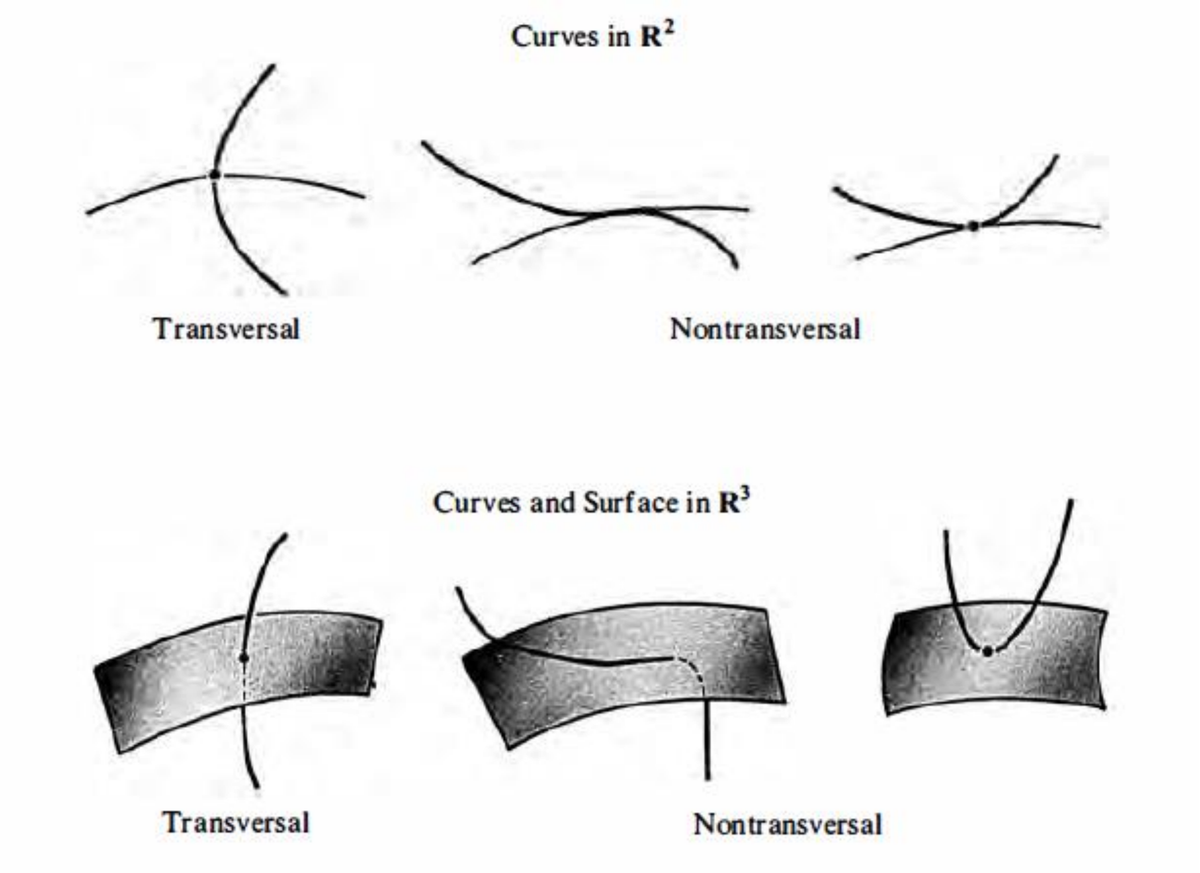
[29] These results have been derived under the assumption that the money market has been ignored. It remains to determine whether the equilibria are in fact supportable as a monetary equilibrium. This question is not addressed here.

The dimensions of the problem then permit the assertion that for $(\omega,k)\in\Omega\times K$ one has that $\chi^{-1}(\omega,k)=(p,\omega,k)\in E$ [30] and for each regular value $(\omega,k)\in\Omega\times K$ there exists an equilibrium price vector $p\in S$. Thus if K has such a dimension and if for each $(\omega,k)\in\Omega\times K$ the set $(p,\omega,k)\in E$ is non-empty then equilibrium exist for every regular value $(\omega,k)\in\Omega\times K$

The question then arises as how to incorporate the space K into the maximisation problems of the two agents. Furthermore, by the introduction of such a contrived parameter space, the interpretation of equilibrium is necessarily altered. Indeed one must then pose the question of whether any resultant equilibria are sunspot or even rational expectations?

Appendix

Figure 1.



[30] Such an assertion would follow from the establishing of the topological degree of the projection map χ to be equal to 1 which would yield propositions analogous to Propositions 1 and 2 above.

Figure 2

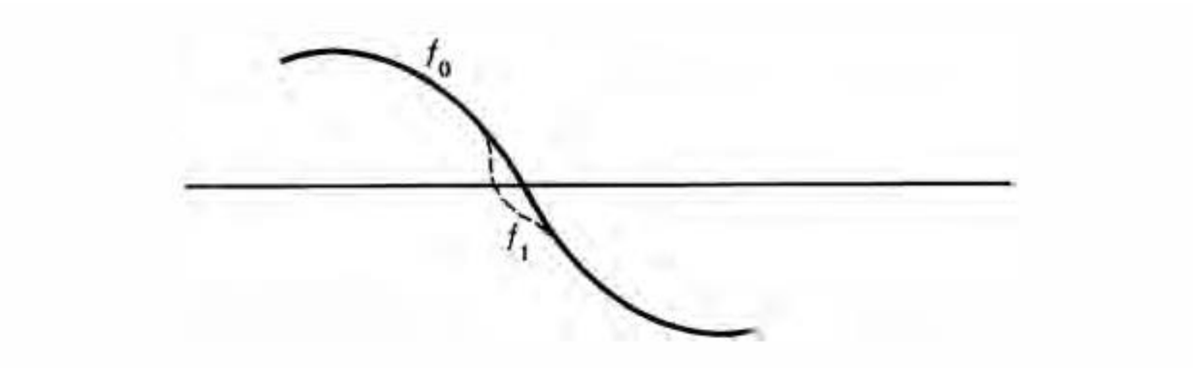
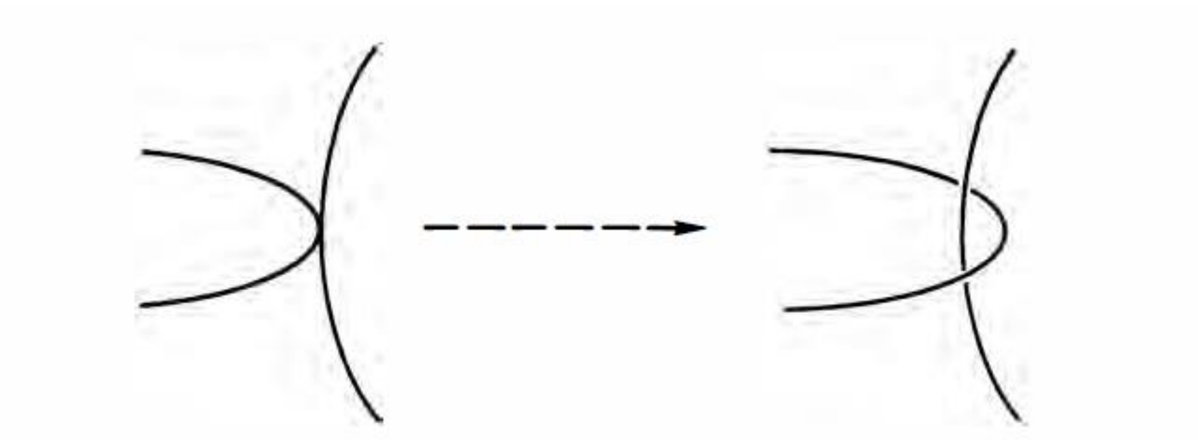


Figure 3.



Figures 1 - 3 are reproduced from Guillemin and Pollack pp. 30 and 35 [11]. In images 1 and 4 of Figure 1 the two geometric objects are transversal to each other. At the point of intersection the two objects are not tangential to each other. In the remaining images of the same figure, it can be seen that at the point of intersection the two objects are tangential to each other and by Definition 1 do not intersect transversally. In the case of non-transversal intersection, transversality is readily obtained by a slight movement in one or the other object (images 2 and 5). It is noted that the dimension of the ambient space is equal to the sum of the dimensions of the two geometric objects. Images 3 and 6 are considered subsequently.

If $X \cap Y = \varnothing$ then by the definition of transversal intersection *X* and *Y* are logically qualified to intersect transversally (Definition 1.i). Consider Figure 2 and suppose that the two objects belong to the space $\mathbb{R}^3$. Then the f_0 intersects the horizontal line *X*. But such intersection is not transversal as a small movement of f_0 to say f_1 suffices to move the curve around *X*. Furthermore, f_0 does not have to move far in order to achieve this thereby f_0 can 'generically' avoid *X*. Hence, $f_0 \pitchfork X \Rightarrow f_0 \cap X = \varnothing$ as stated in Proposition 1 above. Figure 3 illustrates the same principle. Suppose that the two curves belong to $\mathbb{R}^3$ (or any space of higher dimension). Then if the two curves intersect (left hand side of Figure 3) it suffices to perturb one or both of the curves infinitesimally in order to break the property of non-empty intersection (right hand side of Figure 3).

Bibliography

[1] Abraham R. and Robbin J. Transversal Mappings and Flows. W. A. Benjamin Inc. 1967.

[2] Azariardis C. and Guesnerie R. Sunspots and Cycles. Review of Economic Studies, pp. 725 – 737. 1986.

[3] Balasko Y. The Equilibrium Manifold. Postmodern Developments in the Theory of Geneal Economic Equilibrium. The MIT Press. 2009.

[4] Balasko Y. General Equilibrium Theory of Value. Princeton University Press. 2011

[5] Balasko Y., Cass D. and Shell K. Market Participation and Sunspot Equilibria. Review of Economic Studies. Vol. 62. pp. 491 – 512. 1995.

[6] Cass D., and Shell K. Do Sunspots Matter? Journal of Political Economy 91, pp. 193 – 227. 1983.

[7] Davila J., Gottardi P. and Kajii A. Local Sunspot Equilibria Reconsidered. Economic Theory. Vol. 31. pp. 410- 425. 2007.

[8] Golubitsky M. and Guillemin V. Stable Mappings and Their Singularities. Springer Graduate Texts in Mathematics. 1973.

[9] Gottardi P. and Kajii A. The Structure of Sunspot Equilibria: the Role of Multiplicity. Review of Economic Studies. Vol. 69. pp. 713 – 732. 1999.

[10] Guesnerie R. Stationary Sunspots in an *N* Commodity World. Journal of Economic Theory. Vol. 40. Pp 103 – 127. 1986.

[11] Guillemin V. and Pollack A. Differential Topology. Prentice Hall Inc. 1974.

[12] Hirsch M. Differential Topology. 1976. Springer Verlag.

[13] Lee J. M. Introduction to Smooth Manifolds. Springer Graduate Texts in Mathematics. 2002.

[14] Mas-Colell A. The Theory of General Economic Equilibrium. A Differentiable Approach. Cambridge University Press. 1985.

[15] Milnor J. W. Topology from the Differentiable Point of View. 1965. Princeton University Press.

[16] Nagata R. Treatise on Incomplete Asset Markets. Indeterminacy and Inefficiency of Equilibria with Incomplete Asset Markets.

[17] Nagata R. Theory of Regular Economies. Series on Mathematical Economics and Game Theory. Vol. 1. World Scientific Publishing. 2001.

[18] Nagata R. Generic Inefficiency of Equilibria with Incomplete Markets. School of Political Science and Economics. Waseda University.

[19] Peck J. On the Existence of Sunspot Equilibria in an Overlapping Generations Model. Journal of Economic Theory. Vol. 44. pp. 19 – 42. 1988.

[20] Spear S. Sufficient Conditions for the Existence of Sunspot Equilibria. Journal of Economic Theory, 34. pp. 360 – 370. 1984.

[21] Spear S. Rational Expectations in the Overlapping Generations Model. Journal of Economic Theory 35, pp. 251 – 275. 1985.

[22] Tuinstra J. and Weddepohl C. On the Equivalence between the Overlapping Generations Model and Cyclical General Equilibrium Models. Journal of Economics, 70, pp. 187 – 207. 1999.

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