The Dynamics Of Projecting Confidence in Decision Making

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Abstract

We introduce a model in which a decision maker's (DM) projection of confidence in her risk based decisions is enough to generate chaotic dynamics by and through behavioural operations on probability spaces. The model explains preference reversal phenomenon in the context of an ergodic theory of probabilistic risk attitudes and stochastic choice process. We extend the model to fluctuations of the Lyapunov exponent for the behavioural operator in a large sample of DMs with heterogeneous preferences, and we characterize the time dependent probability of chaotic dynamics in that milieu. Specifically, we identify a Perron effect for the empirical Lyapunov exponent process driven by the distribution of DMs heterogeneity. That is, our model predicts that for a seemingly stable system of DMs, *tail event chaos* is triggered by probabilistic optimism and pessimism that fall in a critical range of values for curvature and elevation parameters for subjective probability distributions popularized by behavioural economics and psychology.

Keywords: decision making, confidence, risk, behavioral operator theory, probability theory, dynamical system, chaos

JEL Classification Codes: C02, C44, D03, D81
2010 Mathematics Subject Classification: 37A50, 43A32, 47B34

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1 Introduction

This paper proposes a model of how a decision maker (DM) projects confidence in her risk based decisions, and examines some dynamic consequences of that behaviour. To motivate the model, we note that a decision maker (DM) who purchases a lottery ticket hopes that she will win. She purchases an insurance policy out of fear of loss. In each case, she has a preference for probability distributions over gambling and or insurance. It is known that the cardinal utility function for gambling is convex, and that for insurance is concave. For example, Friedman and Savage (1948) reconciled a DM purchase of insurance and lottery ticket by introducing a utility function with concave and convex segments[1]. See also, Markowitz (1952). In contrast Yaari (1987) duality theory assumes linear utility and shifts all risk attitude to subjective probability weighting[2]. There, DMs overweigh small probabilities (hence optimism about lottery ticket) and underweigh large probabilities (hence pessimism about insured event)[3]. See also Kahneman and Tversky (1979). This phenomenon is usually depicted by an inverse S-shaped, i.e., concave-convex, probability weighting function (PWF)[4]–first plotted by Preston and Baretta (1948)–which cuts the diagonal of a unit square at a de facto fixed point. See Figure 1 on page 2. The DM's hope or optimism about the lottery ticket stems from projecting windfall into a gain domain whereas her fear or pessimism about the insured event stems from projecting catastrophe into a loss domain[5]. These confidence factors are depicted by the areas

[1]A review of the literature shows that Törnqvist (1945) specified a utility function with concave and convex segments and he credited Svennilson (1938) with doing same in another context.

[2]For a simple lottery $(-h, p; h, 1-p)$ at wealth level Y, Arrow (1971, p. 95) represents probabilistic risk attitudes as $p(Y, h) = \frac{1}{2} + \frac{R_A(Y)}{4}h + o(h^2)$ where $R_A(Y)$ is the Arrow-Pratt risk measure decreasing in Y. See also, Merton (1992, p. 218) for util-prob.

[3]Sadiraj (2013) proved that this kind of probabilistic risk attitude suffers from calibration problems that can lead to absurd local risk aversion for extremely small or large probabilities in rank dependent utility (RDU) models like Quiggin (1982); Yaari (1987); and Tversky and Kahneman (1992). See also, Cox et al. (2013).

[4]More recent studies report heterogenous functional forms that range from uniformly convex to uniformly concave with various combinations of concave-convex segments in between. See e.g., Gonzalez and Wu (1999); Abdellaoui (2000); Wilcox (2011); Cavagnaro et al. (2013). Our results are robust to functional forms.

[5]Our use of "projection" is distinguished from the literature on "projection bias" spawned by Loewenstein et al. (2003) which deals with habit formation.

A (optimism) and B (pessimism) in Figure 2. Thus, the psychological space of hope and fear is a separable projective space. See e.g., Shephard (1987); Zhang (2004); He and Zhou (2013). Hence, the geometry of PWFs describes a phase space (or state space) in which DMs transform probability distributions, and project onto state space (gain and loss domains).

Figure 1: Preston and Baretta (1948) psychological probability function

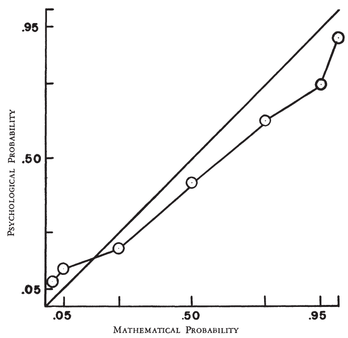
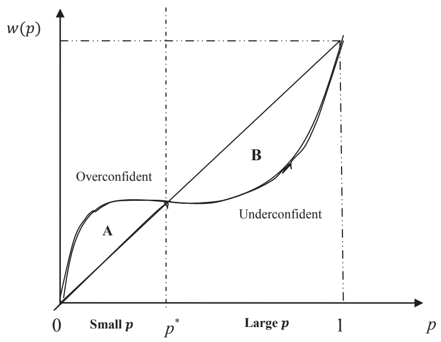


Figure 2: Distribution of Confidence



A seminal paper by Tversky and Wakker (1995) introduced the concept of bounded subadditivity to characterize *what* events impact the probability transformation process. See also, Fox et al. (1996). But they did not provide a constructive model for *how* the transformation takes place. The implicit assumption in all of the papers above is that the PWF is static, i.e., fixed[6]. However, that assumption must be reconciled with preference reversal phenomenon reported in the literature[7]. See e.g., Lichtenstein and Slovic (1973); Tversky (1969). Furthermore, in an experiment designed to test stability of

[6] Andersen et al. (2008, pp. 1111) conducted a field experiment in Denmark and report that "[a]s subjects become more positive about their current finances and more *optimistic* about future expenditures, risk attitudes tend to decline"[emphasis added].

[7] Holt (1986) defines the "preference reversal phenomenon" as one "in which subjects put a lower selling price on the lottery for which they state a preference". Ross (2005, pp. 177-181) reviews the literature on preference reversal and its implications for revealed preference theory. Tversky et al. (1990) attribute the phenomenon to violation of "procedure invariance" which occur when preferences constructed from normative elicitation procedures, which should yield the same results in theory, do not do so in the lab. See also, Smith (1982, pg. 927).

risk attitude over time, Zeisberger et al. (2012) found that risk attitude parameters for prospect theory were stable for aggregate data. However, at the individual level one third of the subjects exhibited instability of risk attitudes over time[8]. See also, Loomes and Sugden (1998); Fox and Tannenbaum (2011). For example, Loomes and Sugden (1998) used *stochastic choice* models to analyse the risk attitude of their subjects over *time*. This suggests the existence of a background driving *stochastic process* induced by time varying stochastic choice. Our model fills a gap in the literature by introducing a behavioural operator that characterizes DMs projection of confidence, parameter instability, and preference reversal over time in state space characterized by a behavioural stochastic process motivated by ergodic theory[9]. Because our model is fairly abstract, it applies to any situation where DMs confidence is in play. For example, in Odean (1998) and Harrison and Kreps (1978) heterogenous DMs are classified by type, and their behaviors are analyzed in a partial equilibrium to explain speculative bubbles. Our model provides a mechanism for classifying such DMs and it shows how probabilistic risk attitudes affect market instability.

Our main result can be summarized as follows. Let Q and P be known probability distributions over loss and gain domains in "mixed frame" state space, see e.g., Harrison and Swarthout (2013), that supports the coexistence of gambling and insurance. We assume that Q represents "small" probabilities and P represents "large" probabilities relative to a fixed point probability $p^\star$. See e.g., Figure 2. In effect, $p^\star$ splits the probability distribution. In our model, beliefs about some parameter θ are accompanied by probability distortions, i.e., PWFs w, reported in the literature on behavioural and experimental economics[10]. See e.g., Fellner (1961); Gonzalez and Wu (1999); Cavagnaro et al. (2013). A DM who has preferences consistent with Von Neumann and Morgenstern (1953) expected utility theory (EUT) will have a linear PWF P, whereas a DM with "nonexpected" utility preferences will have a nonlinear PWF of type $w(P) \neq P$. This feature of our model finds support in the experimental literature. For instance,

[8] Holt (1986, pg. 509) reports that the rate of preference reversal "is typically above 40 percent" in experiments with money incentives.

[9] This result is consistent with violation of the independence axiom due to nonlinearity in probabilities. See e.g., Holt (1986, pg. 511)

[10] Rutström and Wilcox (2009, pg. 621) describe an adaptive learning model introduced by Cheung and Friedman (1997) as a "gamma weighted belief" (GWB) process.

Bruhin et al. (2010) conducted an experiment with a large sample of DMs and report that heterogenous preferences emerged endogenously. Using different methodology, Harrison and Rutström (2009) also find evidence of heterogenous preferences in their mixture model[11]. So that if $P(E)$ is the experimenter's frequentist probability assigned to some measurable event E, when $\theta \in E$, then $w(P(E))$ is the corresponding subjective probability assigned to A. Thus, $w(P(E)) - P(E)$ is a measure of the difference between a DM's subjective probability measure and the experimenter's probability measure for θ in E [12]. $w(P(E)) > P(E)$ implies overconfidence that θ lies in E, whereas $w(P(E)) < P(E)$ implies underconfidence of same[13]. See e.g., A and B in Figure 2. Without loss of generality, we assume that heterogenous [nonlinear] beliefs are of concave-convex type $w(P)$, in contrast to linear beliefs P, even though concave (uniformly optimistic), convex (uniformly pessimistic), convex-concave, and combinations of all of the above have been reported in the literature[14]. See e.g., Gonzalez and Wu (1999); Wilcox (2011); Cavagnaro et al. (2013). Heterogeniety in our model is characterized by random effects superimposed on the concave-convex probability weighting function which is the core theory in our model. See e.g., Loomes and Sugden (1998, pg. 583); Hey and Orme (1994); Barsky et al. (1997); Andersen et al. (2012, pp. 162-163). We exploit the difference in belief to construct a behavioural kernel operator $K(Q,P)$ with domain of definition $\mathcal{D}(K)$. Gemoetrically, the kernel is depicted by areas A and B in Figure 2. There exist functions f and g such that for $f \in \mathcal{D}(K)$,

$$g(Q) = (Kf)(Q) = \int K(Q,P)f(P)dP$$

is the projection of f in Q-space. Thus, if K and g are known, we

[11] Bruhin et al. (2010, pg. 1376) report a $80 : 20\%$ split for $w(P)$ and P, whereas Harrison and Rutström (2009, pg. 146) report a $45 : 55\%$ split. The Bruhin, et al split is rather high. In private communication, Harrison notes that the different splits may be an artifact of the different experimental design and estimation procedures employed in the two papers.

[12] Harrison et al. (2013, Lemma 2, pg. 8) equates this deviation to a function of marginal utility.

[13] Pleskac and Busemeyer (2010, pg. 869) use a probability ratio type statistic to measure confidence.

[14] The curvature and elevation of w reflects the degree of optimism or pessimism and areas of type A and B in Figure 2 extend to any nonlinear PWF. See e.g., Tversky and Wakker (1995); Abdellaoui et al. (2010); Vieider (2012)

can recover $f(Q) = (K^{-1}g)(Q)$ provided the inverse operation K^{-1} is meaningful[15]. In the sequel we show how to construct K and extend it to composite operations that characterize ergodic confidence levels. In our model, beliefs are the only primitives that induce *endogenous* instability.

We extend the behavioural operator K to a large sample of N heterogenous DMs via composite oprtaions $\widehat{T} = -K^T \circ K$. Chaotic dynamics supported by the orbit of $w(P)$ are characterized by an empirical process for the Lyapunov exponent of $\widehat{T}$ induced by the distribution of heterogenous DMs, and estimated from Prelec (1998) 2-parameter probability weighting function. We show how those two parameters–α for curvature and β for elevation–control the probability of *tail event chaos* in an otherwise stable system of DMs. That result is distinguished from Cavalcante, Hugo L. D. de S. et al. (2013) who introduced a model of *tail event chaos* controlled by perturbing the system.

The rest of the paper proceeds as follows. In section 2 we introduce the kernel operator, and report the main results on a dynamical system of confidence. In section 3 we present analytics for the Lyaponov exponent for a large sample of heterogenous DMs. In section 4 we conclude with perspectives.

2 The Behavioural Kernel Operator

The kernel operator is constructed from deviation of subjective probability from an objective probability measure. Let θ be an abstract object in Ω. Our model rests on the following:

Assumption 2.1. *Subjects' prior beliefs about θ can be elicitetd.*

Assumption 2.2. *Prior beliefs about θ are independent.*

Assumption 2.3. *Heterogenous beliefs are of two types: $w(P)$ and P.*

Assumption 2.4. *DMs have preference for probability distributions over ranked outcomes.*

[15] f is not necessarily unique. Furthermore, in the sequel K effectively applies to a Hilbert space so some results may not hold in other spaces. See e.g., Hochstadt (1973, p. 33).

Assumption 2.1 is motivated by Georgia State University (GSU) Credit Risk Officer (CRO) Index which is still in its infancy. There, prior probabilities are elicitetd from a sample of credit risk officers concerning their confidence in the behaviour of 11-major financial market indexes. See Agarwal et al. (2013); Harrison and Phillips (2013).

2.1 Confidence operations over probability domains.

Under Assumption 2.3, p^* is a fixed point probability ($w(p^\star) = p^\star$) that separates loss and gain domains. See e.g., Tversky and Kahneman (1992); Prelec (1998); Cavagnaro et al. (2013). Let $\mathcal{P}_\ell \triangleq [0, p^*]$ and $\mathcal{P}_g \triangleq (p^*, 1]$ be loss and gain probability domains as indicated. So that the entire domain is $\mathcal{P} = \mathcal{P}_\ell \cup \mathcal{P}_g$. Let $w(p)$ be a probability weighting function (PWF), and p be an an objective probability measure.

Definition 2.1 (Behavioural matrix operator)**.**
The confidence index from loss to gain domain is a real valued mapping defined by the kernel function

$$K : \mathcal{P}_\ell \times \mathcal{P}_g \to [-1, 1] \tag{2.1}$$

$$\begin{aligned} K(p_\ell, p_g) &= \int_{p_\ell}^{p_g} [w(p) - p]dp \\ &= \int_{p_\ell}^{p_g} w(p)dp - \frac{1}{2}(p_g^2 - p_\ell^2), \quad (p_\ell, p_g) \in \mathcal{P}_\ell \times \mathcal{P}_g \end{aligned} \tag{2.2}$$

By construction $\int_0^1 \int_0^1 K(q, p)dqdp < \infty$. So K belongs to the Hilbert space of squared integrable functions $L^2([0, 1]^2)$ with respect to Lebesgue measure. See Hochstadt (1973, p. 12). K can be transformed to $\widehat{K}$ so that the latter is singular at the fixed point p^* as follows:

$$\widehat{K}(p_\ell, p_g) = \frac{K(p_\ell, p_g)}{p_g - p_\ell} = \frac{1}{p_g - p_\ell} \int_{p_\ell}^{p_g} w(p)dp - \frac{1}{2}(p_g + p_\ell) \tag{2.3}$$

For internal consistency, we require $K(p_\ell, p_g) = 0, \ p_\ell > p_g$. So K is of Volterra type. See e.g., Hochstadt (1973, pg. 2). Furthermore, $\lim_{p \to p^\star} \widehat{K}(p, p^\star) = \infty$ implies that $\widehat{K}$ is singular near p^* and so $\widehat{K}$ is treated as a distribution. See e.g., Stein (1993, pg. 19). For $\ell = 1, \ldots, m$ and $g = 1, \ldots, r$ $K = \{K(p_\ell, p_g)\}$ is a behavioural matrix operator. □

$\widehat{K}$ is an averaging operator induced by K. It suggests that the *Newtonian potential* or *logarithmic potential* on loss-gain probability domains are admissible kernels. The estimation characteristics of these kernels are outside the scope of this paper. The interested reader is referred to the exposition in Stein (2010).

2.1.1 Extension to stochastic kernels

The kernel above can be extended to probability weighting functions that cut the 45°-line more than once–if at all–by the following means. Let C_o, C_u, $C_\mathfrak{z}$ be the set of probabilities that correspond to overconfidence (o), underconfidence (u) and neutrality ($\mathfrak{z}$). Thus

$$C_o = \{p|\, w(p) > p, p \in [0,1], w : [0,1] \to [0,1]\} \tag{2.4}$$
$$C_u = \{p|\, w(p) < p, p \in [0,1], w : [0,1] \to [0,1]\} \tag{2.5}$$
$$C_\mathfrak{z} = \{p|\, w(p) = p, p \in [0,1], w : [0,1] \to [0,1]\} \tag{2.6}$$

By construction, C_o, C_u, $C_\mathfrak{z}$ are Borel measurable sets, i.e., they are open and monotonic, $C_\mathfrak{z}$ is a zero set, and $\mathcal{P} = \bigcup_j C_j$. Thus, we extend (2.2) to the stochastic kernel:

$$K(p_\ell, C_j) = \int_{p_\ell}^{C_j} (w(p) - p)dp,\ j = 0, u, \mathfrak{z} \tag{2.7}$$

where $K(p_\ell, C_j) = 0$ if $p_\ell > p$ for $p \in C_j$. Underconfidence implies that $\mathrm{sgn}(K(p_\ell, C_j)) = -ve$ and overconfidence implies $\mathrm{sgn}(K(p_\ell, C_j)) = +ve$. If p_ℓ is fixed, then $K(p_\ell, C_j)$ is a set function distributed over C_j. Similarly, for fixed C_j, $K(p_\ell, C_j)$ is a so called Baire function[16] in p_ℓ. Feller (1971, Def. 1, pg. 205) defines a related function for a Markov kernel when K is a probability distribution in C_j. For our purposes, all that is required is that K is measurable which we state in the following.

Lemma 2.5. *The behavioural kernel K is measurable.*

Proof. Since C_j is Borel measurable, and $K(p_\ell, C_j)$ is a Baire function for fixed C_j, K is measurable by virtue of the monotonicity it inherits from Lebesgue integrability over C_j. □

[16] According to Feller (1971, pg. 196) "The smallest closed class of functions containing all continuous functions is called the Baire class and will be denoted by $\mathfrak{B}$. The functions in $\mathfrak{B}$ are called Baire functions". In particular, this class of functions is acceptable as random variables.

The measurability of K implies that the following construct is admissible for measuring behavioural operations. Let $\mathfrak{T}$ be a partiaolly ordered index set on probability domains, and $\mathfrak{T}_\ell$ and $\mathfrak{T}_g$ be subsets of $\mathfrak{T}$ for indexed loss and indexed gain probabilities, respectively. So that

$$\mathfrak{T} = \mathfrak{T}_\ell \cup \mathfrak{T}_g \tag{2.8}$$

For example, for $\ell \in \mathfrak{T}_\ell$ and $g \in \mathfrak{T}_g$ if $\ell = 1, \ldots, m; \quad g = 1, \ldots, r$ the index $\mathfrak{T}$ gives rise to a $m \times r$ matrix operator $K = [K(p_\ell, p_g)]$. Akheizer and Glazman (1961, Pt. I, pp. 54-56) shows how to compute K in the context of Hilbert-Schmidt operator relative to a given orthogonal basis $\{\phi_k(\cdot)\}_{k=1}^{\infty} \in L^2(\mathbb{R})$. The "adjoint matrix"[17] $K^* = [K^*(p_g, p_\ell)] = -[K(p_\ell, p_g)]^T$. So K transforms gain probability domain into loss probability domain–implying fear of loss, or risk aversion, for prior probability p_ℓ. K^* is an *Euclidean motion* that transforms loss probability domain into hope of gain from risk seeking for prior gain probability p_g.

Definition 2.2 (Behavioural operator on loss gain probability domains)**.** Let K be a behavioral operator constructed as in (2.2). Then the adjoint behavioural operator is a rotation and reversal operation represented by $K^* = -K^T$. □

Thus, K^* captures preference reversal phenomenon in probabilistic risk attitudes. Moreover, K and K^* are generated (in part) by prior probability beliefs consistent with Gilboa and Schmeidler (1989) and Kurz (1994). The "axis of spin" induced by this behavioural rotation is along the diagonal of the unit square in the plane in which K and K^* operates in the sequel.

2.2 Ergodic confidence behaviour

Consider the composite behavioural operator $T = K^T \circ K$ and its adjoint $T^* = -T^T = -T$ which is *skew symmetric*. See e.g., Bravo and Pérez (2013, pg. 21).

[17] Technically, the adjoint matrix is defined as $cof(K)^T$ where "cof" means cofactor. See e.g., Strang (1988, pg. 232)

Figure 3: Behavioural operations on probability domains

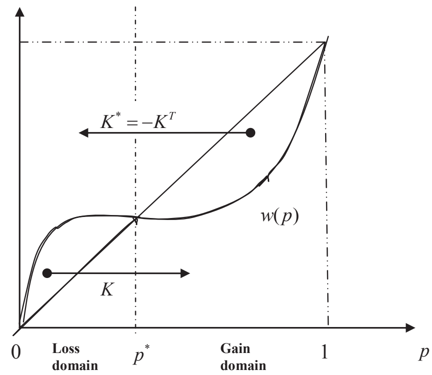
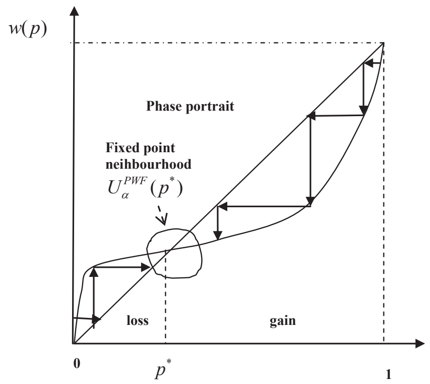


Figure 4: Phase portrait of behavioural orbit



Understanding the adjoint operator T^*

By definition, T^* takes a vector valued function in optimism domain (through K) sends it into pessimism [fear of loss] domain, rotates[18] it and sends it back from a reduced part of pessimism domain (through K^*) where it is transformed into optimism [hope of gain] domain. In other words, T^* is a *contraction mapping* of optimism domain. A DM who continues to have hope of gain in the face of repeated losses in that cycle will be eventually ruined in an invariant subspace which reduces T^*[19]. By the same token, an operator $\widetilde{T}^* = -K \circ K^T = KK^* = -\widetilde{T}$ is a contraction mapping of pessimism domain. In this case, a DM who fears loss of her gains will eventually stop before she looses it all up to an invariant subspace which reduces $\widetilde{T}^*$. Thus, the composite behavior of K and K^* is ergodic because it sends vector valued functions back and forth across loss-gain probability domains in a "3-cycle" while reducing the respective domain in each cycle. These phenomena are depicted on page 9. There, Figure 3 depicts the behavioural operations that transform probability domains. Figure 4 depicts the corresponding phase portrait and a fixed point neighbourhood basis set "centered" at the "attractor" p^*. In what follows, we introduce a behavioural ergodic

[18]This rotation or spin around the diagonal in the unit square is not depicted in Figure 3.

[19]See e.g., Akheizer and Glazman (1961, pg. 82) for technical details on reduced operators.

theory by analyzing T. The analysis for $\tilde{T}$ is similar so it is omitted. Let

$$T = K^T \circ K = K^T K \Rightarrow T^* = -(K^T \circ K)^T = -K^T K = K^* K = -T \tag{2.9}$$

Define the *range* of K by

$$\Delta_K = \{g|\ Kf = g,\ \ f \in \mathcal{D}(K)\} \tag{2.10}$$
$$T^* f = -K^T K f = K^* g \Rightarrow g \in \Delta_K \cap \mathcal{D}(K^*) \tag{2.11}$$
$$\Delta_{T^*} = \{K^* g|\ g \in \Delta_K \cap \mathcal{D}(K^*)\} \subset \mathcal{D}(K^*) \tag{2.12}$$

Thus, T^* *reduces* K^*, i.e. it reduces the domain of K^*, and T is *skew symmetric* by construction.

Lemma 2.6 (Graph of confidence)**.**
Let $\mathcal{D}(K)$, $\mathcal{D}(K^*)$ be the domain of K, and K^* respectively. Furthermore, construct the operator $T = K^*K$. We claim (i) that T is a bounded linear operator, and (ii) that for $f \in \mathcal{D}(K)$ the graph (f, Tf) is closed. □

Proof. See subsection A.1 □

Proposition 2.7 (Ergodic confidence)**.**
Let $\widehat{T} = K^*K$, $f \in \mathcal{D}(T)$ and $\mathcal{D}(K) \cap \mathcal{D}(K^*) \subseteq \mathcal{D}(T)$. Define the reduced space $\mathcal{D}(\widehat{T}) = \{f|\ f \in \mathcal{D}(K) \cap \mathcal{D}(K^*) \subseteq \mathcal{D}(T)$. And let $\mathfrak{B}$ be a Banach-space, i.e. normed linear space, that contains $\mathcal{D}(\widehat{T})$. Let $(\mathfrak{B}, \mathfrak{T}, Q)$ be a probability space, such that Q and $\mathfrak{T}$ is a probability measure and σ-field of Borel measureable subsets, on $\mathfrak{B}$, respectively. We claim that Q is measure preserving, and that the orbit or trajectory of $\widehat{T}$ induces an ergodic component of confidence. □

Proof. See subsection A.2. □

Remark 2.1. Let B be the set of all probabilities p for which $f(p) \in \mathcal{D}(\widehat{T})$. The maximal of such set B is called the *ergodic basin* of Q. See Jost (2005, pg. 141). One of the prerequisites for an ergodic theory is the existence of a Krylov-Bogulyubov type invariant probability measure. See Jost (2005, pg. 139). The phase portrait in Figure Figure 4 on page 9, based on an inverted S-shaped probability weighting function, is an admissible representation of the underlying chaotic behavioural dynamical system. □

Remark 2.2. We note that by construction $\widehat{T} = -K^T K$. So that $\text{sgn}(\widehat{T}) \sim (-)$; $\text{sgn}(\widehat{T}^2) \sim (+)$, and $\text{sgn}(\widehat{T}^3) \sim (-)$ satisfy the "3-period" prerequisite for chaos. See Devaney (1989, pp. 60, 62). Typically, ergodic theorems imply the equivalence of space and time averages. □

Proposition 2.8 (Chaotic behaviour)**.**
The dynamical system $(\mathfrak{B}, \mathfrak{T}, Q, T)$ *induced by confidence in Proposition 2.7 is chaotic because:*

(1) It is sensitive to initial conditions;

(2) It is topologically mixing;

(3) Its periodic orbits are dense.

□

Proof. By hypothesis, ambiguity implies (1). In the proof of Proposition 2.7, it is shown that the orbit generated by the operator $\widehat{T}$ constructed from K and K^* is "3-period" $(-)(+)(-)$. That implies (3) by and through Sarkovskii's Theorem. See Devaney (1989, pp. 60, 62). Moreover, according to Vellekoop and Borglund (1994, pg. 353) and Banks et al. (1992, pg. 332) (1) and (3) $\Rightarrow$ (2). So by definition, the dynamical system induced by the random initial conditions attributed to prior beliefs is weakly chaotic. □

Proposition 2.8 implies that dynamical systems induced by confidence are weakly unpredictable.

2.2.1 Extension to matrix operators

By construction $\widehat{T}$ can be represented by a square matrix operator. In which case, according to Ruelle (1979, Prop. 1.3(b), pg. 30), we have

$$\lim_{n \to \infty} \frac{1}{n} \ln \|\widehat{T}^n u\| = \lambda^{(r)} \tag{2.13}$$

where $\lambda^{(r)}$ is an eigenvalue of $\widehat{T}$, $u \in V^r \backslash V^{(r-1)}$, $r = 1, \ldots, n$ and $V^r = U^{(1)} \oplus \ldots \oplus U^{(r)}$ for eigenspaces U. Thus, (2.13) represents a Lyapunov exponent. See e.g., Walters (1982, pg. 233). Operators like $\widehat{T}$ are typically used to characterize dynamics in linear systems of differential equations of type $\dot{\boldsymbol{x}} = \widehat{T}\boldsymbol{x}$. See, e.g., Arnold (1984, pg. 201). For example, Grasselli and Costa Lima (2012, pg. 206) applied a matrix operator like $\widehat{T}$, to an extended Goodwin-Keene model, to characterize chaotic dynamics in a model of Minsky financial instability hypothesis with so called Ponzi finance. In the following section, we apply (2.13) to characterize chaotic dynamics in a large sample of DMs.

3 Large sample theory of Lyapunov exponents for heterogenous beliefs in finite time

In this section, we consider the *finite-time* behaviour of the Lyapunov exponent for the orbit of subjective probability distributions for a large sample of heterogenous DMs distinguished by a disturbance term. See e.g., Pazó et al. (2013).

3.1 Preliminaries

Definition 3.1 (Lyapunov exponent)**.** Jost (2005, pg. 31). Let $w(p)$ be a probability weighting function such that the first derivative w' exist. The Lyapunov exponent of the orbit $p_n = w(p_{n-1}),\ n \in \mathbb{N}$ for $p = p_0$ is

$$\lambda(p) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \ln |w'(p_j)| \tag{3.1}$$

provided the limit exist.

Assumption 3.1. *The Lyapunov exponent $\lambda(p)$ is in the spectrum of eigenvalues $\sigma(\widehat{T})$ for $\widehat{T}$ in Proposition 2.7.* □

This assumption establishes a nexus between the reduced space $\mathcal{D}(\widehat{T})$ in Proposition 2.7 and the analyses that follow.

Existence of behavioural Feller process

Suppose that starting at time s, a DM transitions from confidence probability w_s at p_ℓ to confidence about a set of probabilities Γ in time t. Let $P\{w_{t+s} \in dy | w_s(p_\ell)\} = K_t(p_\ell, dy)$ be the kernel probability density function for that event. The Chapman-Kolmogorov equation[20] allows us to derive the following transition probability for that DM:

$$P\{w_{t+s} \in \Gamma | w_s(p_\ell\}) = K_{t+s}(p_\ell, \Gamma) = \int_{\mathcal{P}} K_s(p_\ell, dy) K_t(y, \Gamma) \tag{3.2}$$

Subjective transition probabilities of this type are used to characterize heterogenous expectations in Harrison and Kreps (1978, pp. 326-327). Kolmogorov's representation theorem tells us that there exists a behavioural *Feller process* (introduced in (3.12) below) over a filtration $\mathcal{F}_t$ of Brownian motion such that $\{\overline{\lambda}_N(t;p,\alpha,\beta), \mathcal{F}_t; 0 \le t < \infty\}, (\Omega, \mathcal{F}^{\bar{W}}), P^{w(p_\ell)}$ satisfies (3.2). See e.g., Karatzas and Shreve (1991, pp. 292-283). For purposes of exposition, we consider the 2-parameter probability weighting function introduced by Prelec (1998, Prop. 1, pg. 503) and written as

$$w(p) = \exp(-\beta(-\ln(p))^\alpha),\ 0 < \alpha < 1,\ \ \beta > 0 \tag{3.3}$$

By log differentiation we get

$$\ln[w'(p)] = \ln(\alpha\beta) + (\alpha - 1)\ln(-\ln(p)) - \ln(p) - \beta(-\ln(p))^\alpha \tag{3.4}$$

Thus, the behavioural operator in Definition 2.1 can be written as $K(\cdot;\alpha,\beta)$ where α is a curvature parameter and β is an elevation parameter–each of which controls confidence levels. See e.g., Abdellaoui et al. (2010). Monotonicity of $w(p)$ guarantees that $w'(p) > 0$ so the absolute value requirement in (3.1) is satisfied. However, the true probability weighting function $w(p)$ is unknown, so the parameters α and β are unobservable in phase space.

Assumption 3.2. *Heterogenous DMs are distributed according to* $\epsilon_i \sim iid(0,\sigma^2),\ \ i = 1,\ldots,N$. □

Consider a large sample of N heterogenous decision makers (DMs). DMs [unobserved] heterogeneity is accounted for by appending ϵ_i to (3.4)[21]. See e.g., Hey and Orme (1994); Barsky et al. (1997); Harrison and Rutström (2008) and Zeisberger et al. (2012). In which case the subjective probability

$$\ln[w'(p)]^o = \ln[w'(p)] + \epsilon_i \tag{3.5}$$

[20]See Feller (1971, pg. 351)

[21]We could model heterogeneity as a random effect such as $\alpha_i = \mu_i + \epsilon_{ij}$ where $\mu_i \sim (0,\sigma^2)$ and ϵ_{ij} is a "treatment effect" or news. See e.g., (Kutner et al., 2005, pg. 1031).

is an observed (o) random coefficient, i.e., random variable. See e.g., Andersen et al. (2012, pg. 162). This is important because by definition a collection of random variables observed at different points in time is a sample function of a stochastic process. In Lian and Stenlund (2012, pp. 241-242) the evolution of an object like $\ln[w'(p)]^o$ was presented in the context of a dynamical system–which we do in the sequel. Let $[0, T]$ be the finite time interval for which Lyapunov exponents are observed for each DM. Without loss of generality we normalize the time interval to coincide with $[0, 1]$ and let $\Pi^{(n)} = \{0, t_1^{(n)}, t_2^{(n)}, \dots, t_k^{(n)}, \dots, 1\}$ partition $[0, 1]$ into dyadic intervals such that $t_k^{(n)} = k.2^{-n}$. This facilitates analysis in the function space $C[0, 1]$. So the heterogeneous subjective probability in (3.5), observed at each time point $t_k^{(n)}$, $k = 1, \dots, 2^n$, is distinguished by the heterogeneity factor $\epsilon_i(t_k^{(n)})$.

3.2 Chaos and fluctuation of empirical Lyapunov exponent process

Consider the cumulative effect of heterogeneity factors at time $t \in [t_k^{(n)}, t_{k+1}^{(n)})$ defined by

$$S_{nt}^j = \sum_{k=1}^{nt} \epsilon_j(t_k^{(n)}), \qquad S_{[nt]}^j = \sum_{k=1}^{[nt]} \epsilon_j(t_k^{(n)}) \tag{3.6}$$

where $[nt]$ is the integer part of nt for arbitrary n. The random broken line connecting the points $([nt], S_{[nt]}^j)$ and (nt, S_{nt}^j) is given by the empirical process

$$W_n^j(t) = S_{[nt]}^j + (nt - [nt])\epsilon_j([nt] + 1) \tag{3.7}$$

By virtue of Donsker's Theorem, we assume that $W_n^j(t)$ is an approximate Brownian motion in the space of continuous functions $C[0, 1]$. See e.g., Serfling (1980, p. 41); Knight (1962); Shorack and Wellner (1986, pp. 52-53); Karatzas and Shreve (1991, pg. 66). Let $w(t; p)$ be the state of the PWF at time t. According to Gikhman and Skorokhod (1969, pp. 370-371), by virtue of (3.4), (3.5) and (3.7) the incremental change in time dependent PWF for the j-th DM at time t can be written as

$$\Delta \ln[w'^j(t;p)] = a^j(p;\alpha,\beta)\Delta t + \sigma \Delta W_n^j(t), \quad \text{for constant drift} \tag{3.8}$$

$$a^j(p;\alpha,\beta) = \ln(\alpha\beta) + (\alpha-1)\ln(-\ln(p)) - \ln(p) - \beta(-\ln(p))^\alpha \tag{3.9}$$

For example, $a^j(p;\alpha,\beta)$ is the constant drift term comprised of probabilistic risk attitude factors in a stochastic differential equation, characterised by an unobserved background driving empirical process $W_n^j(t)$, induced by heterogeneous DMs stochastic choice measured at different points in time. Thus, the aggregate change in PWF for the sample size N is given by

$$\sum_{j=1}^{N} \Delta \ln[w'^j(t;p)] = \sum_{j=1}^{N} a^j(p;\alpha,\beta)\Delta t + \sigma \sum_{j=1}^{N} \Delta W_n^j(t) \tag{3.10}$$

Substituting $\Delta \ln[w'^j(t;p)]$ for $\ln|w'(p_j)|$, $j = 1,\ldots,m$ in (3.1)[22], and replacing Δt and ΔW_n with dt and dW_n respectively gives us

$$\sum_{j=1}^{N} d\lambda^j(t;p,\alpha,\beta) = \frac{1}{m}\sum_{j=1}^{N}\sum_{r=1}^{m} a^j(p;\alpha,\beta)dt + \frac{1}{m}\sum_{j=1}^{N}\sum_{r=1}^{m} dW_n^j(t) \tag{3.11}$$

Dividing LHS and RHS by N and using "bar" to represent sample average, we get the stochastic Lyapunov exponent process

$$d\bar{\lambda}_N(t;p,\alpha,\beta) = \bar{a}_{m,N}(p;\alpha,\beta)dt + \sigma d\overline{W}_{n,N}(t), \tag{3.12}$$

$$\bar{\lambda}_N(\cdot) = \frac{1}{N}\sum_{j=1}^{N}\lambda^j(\cdot), \quad \bar{a}_{m,N}(\cdot) = \frac{1}{N}\sum_{j=1}^{N} a^j(p;\alpha,\beta), \quad \overline{W}_{n,N}(t) = \frac{1}{N}\sum_{j=1}^{N} W_n^j(t) \tag{3.13}$$

$\overline{W}_{n,N}(t)$ is the background driving Brownian motion induced by stochastic choices of heterogeneous DMs. Integrating the stochastic differential equation in (3.12) gives us the following.

[22] $w^j(t;p)$ is the weighted probability $w(p_j)$ at time t, with initial value at p

Stability condition

We note that Leonov and Kuznetsov (2007, Def. 9) defines a *large time* Lyapounov exponent as $\limsup_{t\to\infty} \frac{1}{t} \ln \lambda(t)$. However, our analysis is based on finite-time. The stability condition, see e.g., Leonov and Kuznetsov (2007), corresponds to negative eigenvalues in our model and is given by

$$\sup_t \bar{\lambda}_N(t;p,\alpha,\beta) < 0 \Rightarrow \sigma \sup_t \left(\overline{W}_{n,N}(t) - \overline{W}_{n,N}(0)\right) + \int_0^t \bar{a}_{m,N}(p;\alpha,\beta)du < 0 \tag{3.14}$$

$$\Rightarrow \sup_t \overline{W}_{n,N}(t) < \overline{W}_{n,N}(0) - \frac{1}{\sigma}\bar{a}_{m,N}(p;\alpha,\beta)t, \quad \overline{W}_{n,N}(0) = 0 \tag{3.15}$$

3.3 Estimating the behavioural probability of chaos

To estimate the probability of stability we rewrite $\overline{W}_{n,N}(t)$ in (3.13) as an approximate Brownian motion

$$\overline{W}_{n,N}(t) \equiv W_n\left(\frac{t}{N}\right) \tag{3.16}$$

If $\phi(\cdot)$ is the probability density function for $\overline{W}_{n,N}(t)$, then the probability density function for $\sup_t \overline{W}_{n,N}(t)$ is proportional to $\phi(\cdot)$. See e.g., Gikhman and Skorokhod (1969, pg. 286); Karatzas and Shreve (1991, pg. 96, Prob. 8.2). So that

$$\Pr\left\{\sup_t \overline{W}_{n,N}(t) < -\frac{1}{\sigma}\bar{a}_{m,N}(p;\alpha,\beta)t\right\} = \Pr\left\{\sup_t W_n\left(\frac{t}{N}\right) < -\frac{1}{\sigma}\bar{a}_{m,N}(p;\alpha,\beta)t\right\} \tag{3.17}$$

$$= c_0\Phi\left(-\frac{\bar{a}_{m,N}(p;\alpha,\beta)}{\sigma}\sqrt{Nt}\right) = \varphi(t,\alpha,\beta,N,\sigma) \tag{3.18}$$

where c_0 is a constant of proportionality, $\Phi(\cdot)$ is the cumulative normal distribution and $\varphi(\cdot)$ is a numerical probability. Thus, DMs heterogeneity, $\overline{W}_{n,N}(t)$, induces a *Perron effect* with *tail event probability of chaos*

$$\Pr\left\{\sup_t \overline{W}_{n,N}(t) \geq -\frac{1}{\sigma}\bar{a}_{m,N}(p;\alpha,\beta)t\right\} = 1 - \varphi(t,\alpha,\beta,N,\sigma) \tag{3.19}$$

in a seemingly stable system. See e.g., Leonov and Kuznetsov (2007). According to (3.18) given α, β, σ at time t, the probability of instability in (3.19) increases as N gets larger. To evaluate the impact of the other control variables on the probability of instability in (3.19), we turn to comparative statics. Rewrite the drift term $a^j(p; \alpha, \beta)$ in (3.9), which is the same in (3.13) for given p, as

$$f(\alpha, \beta; p) = \ln(\alpha\beta) + (\alpha - 1)\ln(-\ln(p)) - \ln(p) - \beta(-\ln(p))^{\alpha} \tag{3.20}$$

$$\frac{\partial f}{\partial \alpha} = \alpha^{-1} + \ln(-\ln(p)) - \beta(-\ln(p))^{\alpha+1} \tag{3.21}$$

$$\frac{\partial f}{\partial \alpha} > 0 \Rightarrow 0 < \beta < \frac{\frac{1}{\alpha} + \ln(-\ln(p))}{(-\ln(p))^{\alpha+1}} \tag{3.22}$$

Similarly,

$$\frac{\partial f}{\partial \beta} = \frac{1}{\beta} - (-\ln(p))^{\alpha}, \quad \frac{\partial f}{\partial \beta} > 0 \quad \Rightarrow 0 < \beta < (-\ln(p))^{-\alpha} \tag{3.23}$$

The first order effects for increasing drift (and hence increased probability of instability in (3.19)) in (3.22) and (3.23) is given by

$$0 < \beta < \min\left\{\frac{\alpha^{-1} + \ln(-\ln(p))}{(-\ln(p))^{\alpha+1}}, \quad (-\ln(p))^{-\alpha}\right\} \tag{3.24}$$

Since α controls the curvature of $w(P)$, it determines the degree of DM's confidence. So (3.24) depicts the range of elevated confidence that support an increased likelihood of chaos. In the case of Prelec (1998) single factor model, i.e., $\beta = 1, 0 < \alpha < 1$, we find that the set of feasible values in (3.24) for curvature α are solutions to the nonlinear equation

$$\frac{\alpha^{-1} + \ln(-\ln(p))}{(-\ln(p))^{\alpha+1}} > 1 \tag{3.25}$$

$$\Rightarrow (-\ln(p))^{\alpha+1} - \alpha^{-1} - \ln(-\ln(p)) < 0 \tag{3.26}$$

The solution (if it exists) to (3.26)suggests that there are at most countably many DMs of type α who would induce chaotic dynamics by preference reversal. Thus, (3.26) allows us to classify DMs who are prone to exhibit chaotic behaviour. We summarize the result above[23] in

[23]These results also appear in Cadogan-Charles (2014)

Proposition 3.3 (Chaotic probabilistic risk attitudes)**.** *Given a large sample of heterogenous DMs with Prelec (1998) 2-parameter PWFs (α and β) in a dynamical system of projecting confidence in psychological space, the tail event probability $1-\varphi$ that the system becomes chaotic depends on either of the following*

1. *growth in sample size N;*
2. *risk attitude parameters α (curvature) and β (elevation) that induce the range of confidence*

$$0 < \beta < \min\left\{\frac{\alpha^{-1}+\ln(-\ln(p))}{(-\ln(p))^{\alpha+1}},\ \ (-\ln(p))^{-\alpha}\right\}$$

3. *increased precision in σ for classifying heterogenous DMs.*

Proposition 3.4 (Classifying chaotic decision makers)**.** *A decision maker with Prelec (1998) 1-parameter PWF exhibits chaotic behavior if the type α curvature of her subjective probability distribution satisfies the inequality*

$$(-\ln(p))^{\alpha+1} - \alpha^{-1} - \ln(-\ln(p)) < 0$$

□

If $\alpha_r,\ r = 1,\ldots,K$ are solutions to (3.26), then Proposition 3.4 constitutes a classification scheme. It says that there are $\alpha_1,\ldots,\alpha_K$ probabilistic risk attitude measures that are susceptible to preference reversal and *tail event chaos* dynamics. Thus, given p it is possible to derive numerical estimates for α_r and test experimentally whether the classification scheme holds.

4 Conclusion

We introduced a model in which decision makers (DMs) projection of confidence is sufficient to generate chaotic dynamics by and through a behavioural kernel operator that transforms probability spaces. We extend the model to fluctuations for the Lyapounov exponent in a large sample of DMs with heterogenous beliefs, and we characterize the time dependent probability of chaotic dynamics in that milieu. Specifically, we show how probabilistic risk attitude factors for optimism and pessimism control the chaotic dynamics in a large system of DMs. In particular, our theory identifies the critical range of curvature and elevation parameter values that support preference reversal. Our results have implications for credit markets where confidence and risk management play a key role. For example, according to the Georgia State University, CRO Risk Index website[24] "[t]he CRO Risk Index seeks to aggregate the subjective opinions of global risk professionals regarding significant movements in financial markets and general economic conditions". This paper provides an analytic framework for characterizing the role of "subjective opinions" in those markets by and through a 2-parameter subjective probability distribution. The model presented here can be extended to accommodate different behavioural stochastic processes for the Lyapunov exponent. For example, adding random effects to either the curvature or elevation parameters (or both) would affect the drift term in the behavioural stochastic process.

A Appendix of Proofs

A.1 Proof of Lemma 2.6–Graph of confidence

Proof.

(i). That T is a bounded operator follows from the facts that the fixed point p^* induces singularity in K and K^*. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of operators induced by an appropriate corresponding sequence of $K_n{'s}$ and $K_n^{*}{'s}$, and $\sigma(T_n)$ be the spectrum of T. Thus, we write $\|T_n\| = \prod_{j=1}^{N_n} \lambda_j,\ \ \lambda_j \in \sigma(T_n)$, where $N_n = \dim \sigma(T_n)$. Singularity implies $\lim_{N_n\to\infty} \lambda_{N_n} = 0$ and for $\lambda \in \sigma(T)$, we have $\lim_{n\to\infty} \|T_n - T\| \leq \lim_{n\to\infty} |\lambda_n - \lambda| \|f\| = 0$. Thus, $T_n \to T$ is bounded.

(ii). Let $f \in \mathcal{D}(K)$ and $C(x) = (Kf)(x)$. So $(Tf)(x) = (K^*Kf)(x) = (K^*f^*)(x) = C^*(x)$ for $f \in \mathcal{D}(T)$. For that operation to be meaningful we must have $f^* \in \mathcal{D}(K^*)$. But $T^* = -T^T = K^TK = -T \Rightarrow f^* \in \mathcal{D}(T)$. According to the Open Mapping Closed

[24] See http://www.gsucroriskindex.org/. Last visited 2014/02/18

Graph Theorem, see Yosida (1980, pg. 73), the boundedness of T guarantees that the graph $(f, Tf) \in \mathcal{D}(T) \times \mathcal{D}(T^*)$ is closed.

□

A.2 Proof of Proposition 2.7–Ergodic confidence

Proof. Let $f \in \mathcal{D}(\widehat{T})$. Then $(\widehat{T}f)(x) = (K^*Kf)(x) = (K^*f^*)(x) = C^*$ for $f^* \in \mathcal{D}(T^*)$. But $T^* = -T^T = -(-K^TK) = K^TK = -T \Rightarrow f^* \in \mathcal{D}(T)$. Since f is arbitrary, then by our reduced space hypothesis, $\widehat{T}$ maps arbitrary points f in its domain back into that domain. So that $\widehat{T} : \mathcal{D}(\widehat{T}) \to \mathcal{D}(\widehat{T})$. Whereupon from our probability space on Banach space hypothesis, for some measureable set $A \in \mathfrak{T}$ we have the set function $\widehat{T}(A) = A \Rightarrow \widehat{T}^{-1}(A) = A$ and $Q(\widehat{T}^{-1}(A)) = Q(A)$. In which case $\widehat{T}$ is measure preserving. Now by Lemma 2.6, $(\widehat{T}C^*)(x) = \widehat{T}(\widehat{T}f)(x) = (\widehat{T}^2f)(x) \Rightarrow (f, \widehat{T}^2f)$ is a closed graph on $\mathcal{D}(\widehat{T}) \times \mathcal{D}(\widehat{T}^*)$. By the method of induction, $(f, \widehat{T}^jf),\ j = 1, \dots$ is also a graph. In which case the evolution of the graph $(f, \widehat{T}^jf),\ j = 1, \dots$ is a dynamical system, see Devaney (1989, pg. 2), that traces the trajectory or orbit of f. Now we construct a sum of N graphs and take their average to get

$$f_N^*(x) = \frac{1}{N}\sum_{j=1}^{N}(\widehat{T}^jf)(x) \tag{A.1}$$

According to Birchoff-Khinchin Ergodic Theorem, Gikhman and Skorokhod (1969, pg. 127), since Q is measure preserving on $\mathfrak{T}$, we have

$$\lim_{N\to\infty} f_N^*(x) = \lim_{N\to} \frac{1}{N}\sum_{j=1}^{N}(\widehat{T}^jf)(x) = f^*(x) \ \ a.s.\ Q \tag{A.2}$$

Furthermore, f^* is $\widehat{T}$-invariant and Q integrable, i.e.

$$(\widehat{T}f^*)(x) = f^*(x) \tag{A.3}$$

$$E[f^*(x)] = \int f^*(x)dQ(x) = \lim_{N\to} \frac{1}{N}\sum_{j=1}^{N} \int (\widehat{T}^j f)(x)dQ(x) \tag{A.4}$$

$$= \lim_{N\to} \frac{1}{N}\sum_{j=1}^{N} E[\widehat{T}^j f)(x)] \tag{A.5}$$

Moreover,

$$E[f^*(x)] = E[f(x)] \Rightarrow (\widehat{T}E[f^*(x)]) = \widehat{T}E[f(x)] = E[(\widehat{T}f)(x)] = E[C(x)] \tag{A.6}$$

So the "time average" in (A.5) is equal to the "space average" in (A.6). Whence $f \in \mathcal{D}(\widehat{T})$ induces an ergodic component of confidence $C(x)$. $\square$

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