Discrete Closed-Form Solutions for Barrier Options

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Abstract

Pricing barrier options in discrete-time using lattice techniques is not a straightforward exercise. While the use of larger numbers of time steps may produce more accurate values for standard options, the value of a barrier option is extremely sensitive to the number of time steps used. Merely using more time steps will often produce erroneous option values while simultaneously increasing computation time. There is a consequential necessity for closed-form solutions for this class of derivative. This paper outlines how a reformulation of a simple random walk and an application of the reflection principle may be used to find a binomial coefficient that counts the number of ways a sample path will cross a constant barrier. In this way a general discrete closed-form solution, analogous to Cox, Ross and Rubinstein’s result, may be found for some barrier options. The case of a down-and-in call option is examined in detail. In addition, we prove the convergence of this discrete solution to its continuous-time counterpart.

1 Introduction

An extremely brief introduction to the theory of option pricing follows. There are a number of extremely good technical and financial review articles available. The interested reader is urged to consult Bensoussan (1984) and Davis (2001), and the many references therein, for extensive details.

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The core idea behind the celebrated option pricing formula of Black and Scholes (1973) is to be found in a slightly different form in the work of Merton (1973). The option price at any time in its life may be replicated or mimicked by an equivalently valued portfolio—the hedge/replicating portfolio. In general, this portfolio consists of a variable position in the risky, reference asset and a riskless investment (bond). A strategy that ensures both the exact replication of the option and the elimination of the possibility of arbitrage yields the surprising results of self-financing and objective pricing. The self-financing property implies that the premium paid up front for the option is exactly sufficient for funding the replication strategy and no additional inflow of cash is required during the option’s life. The more surprising result of objective pricing implies that individual investor views of the expected return on the reference asset play no role at all in the fair price of the option. Both of these results are dependent on the asset price process chosen (i.e. the mathematical model of price movements) and the regularity of the resulting option price function. In discrete-time, this coincides with assuming that the asset price process is binomial, and in continuous-time, in coincides with the well-known assumption of log-normally distributed future values for the stock price.

Nearly ten years after the fact, the framework for option pricing was rewritten and reformulated by Harrison & Pliska (1981) and Harrison & Kreps (1979). The problem of option pricing can be expressed in terms of martingale theory. Martingales are particular random processes associated with “fair games”, where the expected profit is equal to zero. Martingale pricing theory allows a much deeper level of mathematical sophistication in the option pricing problem. Furthermore, the martingale framework is the natural and sufficient description for derivative pricing.

Exotic options are a generic name given to derivative securities that have more complicated cashflows or payoffs than are associated with “vanilla” puts and calls. Barrier options are path dependent exotic options. They are similar to standard options except that they are extinguished or activated only when the underlying asset price reaches a predetermined boundary or barrier price. The most obvious reason for the popularity of these exotic options is that they are cheaper than vanilla options. Also, their payoff structure provides increased flexibility to hedgers. Analytical solutions exist for most simple barrier options, and their valuation is described in Rubenstein and Reiner (1991), for example. Ideally one would want to find Black-Scholes type solutions for all barrier options. However, as the
payoff structure of the barrier options becomes more complicated, so does the difficulty in finding closed-form solutions. In most cases closed-form solutions do not exist, e.g. American barrier options and powered barrier options, and these must be valued by using numerical methods. There is a further, more subtle, reason why continuous-time solutions for barrier options may incorrectly value the option. Continuous-time solutions assume that the option is triggered as soon as the barrier is breached. In reality, most barrier options are checked discretely (in general, once a day with respect to the closing price) which suggests that a discrete-time solution to the pricing problem is more suitable.

Barrier options with a single, non-time-dependent, one-dimensional barrier will be referred to here as “vanilla” barrier options. “Vanilla” barrier options are either Knock-in (or In) Options or Knock-out (or Out) Options. Knock-in and Knock-out Options may be further categorised by the position of the barrier relative to the initial value of the underlying asset. If the barrier is below the underlying price at contract inception the option is referred to as a down barrier option. If the barrier is above the underlying price at contract inception the option is referred to as an up barrier option. Other kinds of barrier options include options that have two barrier levels, a time-dependent barrier level or a more complicated option whose payoff is dependent on one underlying and whose barrier is dependent on another underlying (two-dimensional barrier options). Collectively these options form the most popular path dependent equity options traded internationally and in the OTC market Kou (2001).

This paper focuses on the valuation of European barrier options by using binomial lattices. Section 2 discusses the problems involved in a straightforward application of lattice methods in pricing barrier options. Although a number of lattice methods applied to the problem of pricing European barrier options have been proposed, see Boyle and Lau (1994), Derman et al (1995), Figlewski and Gao (1998) and Ritchken (1995), none of them includes a closed-form solution. Kou (2001) proposes a modification to known continuous-time solutions which incorporates a discrete monitoring of the barrier.

The solution to all lattice methods previously proposed requires stepwise recursive discounting of the payoffs at maturity to contract inception. This technique does not facilitate a straightforward proof of a closed-form solution as the limiting case of the binomial lattice. An approach which allows the derivative price to be written with the path dependence
embedded in the marginal values is consequently more tractable.

Section 3 develops, in rigorous detail, a closed-form solution for the pricing of barrier options in an analogous way to that proposed by Cox, Ross and Rubenstein (1979) for evaluating standard puts and calls. Section 4 discusses the particular case of vanilla barrier options. It is shown in this case, that the closed-form solution of section 3 converges to the continuous-time solution. We include some numerical results and comparisons.

2 Lattice Methods for Barrier Options

Binomial trees are usually simple to implement and understand, and are intuitively powerful tools in the valuation of option prices. The implicit assumption in the binomial model of option pricing is that there is a continuous-time model of stock price movements to which the discrete model is an approximation, see Chriss (1979). Usually, binomial trees are used to price options for which no analytical solution exists. In addition, the same trees should be able to reproduce existent continuous-time model prices to some pre-specified accuracy. Any lattice method suffers from at least two sources of inaccuracy Derman et al (1995) identifies: “Stock Price Quantisation Error” and “Option Specification Error”.

Stock Price Quantisation Error is a result of the discretisation of time, resulting in a lattice of discrete stock prices. Valuing options on such a lattice is essentially the same as valuing an option on a discretely moving stock. Clearly this modelling approach fails to capture the continuous-time nature of stock price movements and results in “quantisation error”. To remedy the error, a lattice with an infinitesimal mesh should be used. In practice a sufficiently large number of time steps is chosen such that increasing the number of time steps has a negligible effect on the price accuracy.

Option Specification Error is the result of the failure of the lattice to accurately represent the terms of the option. Option specification error is not a problem when valuing European options (where only the marginal distribution is of interest) but exotic options (particularly barrier options) are very sensitive to this type of error.

In theory, barrier options can be valued using the binomial tree in a conventional way as described in Cox, Ross and Rubenstein (1979). However, a naive application of the simple binomial method is inappropriate for pricing barrier options. The use of an optimal number
of time steps that gives accurate results for vanilla options may give inaccurate results for barrier options, see Chriss (1979), Derman et al (1995). This is because barrier options are exposed to both “quantisation error” and “option specification error”.

The price of a barrier option is sensitively dependent on the location of the barrier within the lattice. This sensitivity is a result of the non-smooth nature of the option value near the boundary. If the barrier falls between two layers of nodes the horizontal barrier is effectively shifted to a nearby layer of nodes. Hull (2000) defines the inner barrier as the barrier formed by nodes on the inside of the true barrier and the outer barrier as the barrier formed by nodes just outside the true barrier. A binomial tree assumes the outer barrier to be the true barrier because the barrier conditions are first used by these nodes. This causes “option specification error” where the option is priced at a different barrier to the one specified by the contract. The tree has failed to capture the terms of the contract. This results in slow convergence and requires a large number of time steps to obtain a result which is persistently biased above or below the correct value.

Boyle and Lau (1994) suggest using the standard binomial model of Cox, Ross and Rubenstein (1979) but constrained in such a way that the lattice has a layer of nodes as close as possible to the barrier. They derive a relationship between the number of lattice time steps \( n \) and the accuracy with which the barrier can be represented on the lattice, and deduce the optimal lattice size from this relationship. Under conditions of optimality, the barrier implied by the tree will be very close to the true barrier eliminating most of the option specification error. (This method breaks down if more than one barrier is present or the barriers are time-varying.)

3 The Closed-Form Solution

The formalism and notation of this section follows that of Cox, Ross and Rubenstein (1979) (CRR). In particular, denote by \( T \) the maturity of the derivative under consideration and divide the time interval \([0, T]\) into \( n \) intervals of equal length \( \frac{T}{n} \). If \( S \) denotes a state of the stock price at time step \( i \), \( 0 \leq i < n \), at time step \( i+1 \) two possible states are attainable: \( uS \), resulting from an ‘up’ movement, or \( dS \) resulting from a ‘down’ movement. Here \( u = e^{\sigma \sqrt{\frac{T}{n}}} \) and \( d = 1/u = e^{-\sigma \sqrt{\frac{T}{n}}} \), where \( \sigma \) is the volatility of the stock price per unit time period. Denoting by \( r \) the constant risk-free rate of return over \([0, T]\), compounded continuously
1, σ and r are assumed to obey $d < e^{rT/n} < u$. Then, if $S_T$ represents the stock price at time $T$ (that is, at time step $n$), $S_T$ may take one of $n + 1$ possible values: $S_0u^jd^{n-j}$ for $j = 0, \ldots, n$. Note that only the number and not the sequence of up and down moves determines the value of $S_T$. Each permutation of up and down moves is known as a path in the lattice. Letting $H(S_T)$ be the payoff at $T$ of a (European) derivative, with underlying $S$, the CRR price formula for the derivative is

$$e^{-rT}\sum_{j=0}^{n} C^n_j p^j(1-p)^{n-j}H(S_0u^jd^{n-j})$$

where $C^n_j$ represents the number of paths terminating at $S_0u^jd^{n-j}$ (which is $\binom{n}{j}$ for a vanilla option) and $p$ is the risk-neutral probability of an ‘up’ movement.

For a down-and-in option the price of the underlying must touch or cross the barrier, $B$ (which lies below $S_0$), on or before $T$ to receive the payoff ($H(S_T)$) at $T$.

Given $n$, let $m$ satisfy

$$S_0d^m \leq B < S_0d^{m-1}$$

$$\Leftrightarrow m \geq \frac{\log \left( \frac{B}{S_0} \right)}{\log d} > m - 1$$

Then define

$$m = \left\lceil \frac{\log \left( \frac{B}{S_0} \right)}{\log d} \right\rceil = \frac{\log \left( \frac{B}{S_0} \right)}{\log d} + \varepsilon \quad \varepsilon \in [0, 1) \text{ constant.}$$

Now, for each $j$, there are $\binom{n}{j}$ possible paths that reach $S_0u^jd^{n-j}$. Of these paths only some will cross the barrier. Which paths cross the barrier depends on both the number and the order in which the down movements appear in the permutations of the $j$ up movements and $n-j$ down movements. The problem is to determine for each $j$ how many of the paths that terminate at $S_0u^jd^{n-j}$ cross the barrier. By construction, a path crosses the barrier if and only if at some time step $n_m$ ($0 \leq n_m \leq n$) the net number of down movements at $n_m$ (that is, the number of down movements on or before $n_m$ minus the number of up movements on or before $n_m$) equals or exceeds $m$.

If $j$ is the total number of up movements at time step $n$, the net number of down movements at maturity is $z = n - 2j$. The value of $z$ relative to $m$ is of crucial importance in

\footnote{This representation of the risk-free rate differs from Cox, Ross and Rubenstein (1979)}
determining the appropriate number of paths which have crossed the barrier.

Consider first the case $z \geq m$: As $z$ lies above $m$ every path with $z$ net down movements has, on or before time step $n$, at least $m$ net down movements (and so the corresponding path in the lattice has touched or crossed the barrier). So, the number of $\binom{n}{m}$ paths with $m$ down movements on or before time step $n$ and $z$ net down movements at time step $n$ equals the number of paths with $z$ net down movements at time step $n$,

$$\left( \frac{n}{\frac{1}{2}(n + z)} \right) = \binom{n}{n - j} = \binom{n}{j}$$

Now consider the case $z < m$: The appropriate number of paths is determined by applying the reflection principle. Refer to Figure 1 below:

![Figure 1: The Reflection Principle](image)

Consider a path that breaches the barrier at time $n_m$ (i.e. the path has $m$ net down movements at time step $n_m$) and has $z$ net down moves at time step $n$. By reflection, there is a quasi-symmetrical path with $m$ net down movements at time step $n_m$ and $2m - z$ net down movements at time step $n$. As the path considered is arbitrary this reflection applies to all paths: the number of paths with $m$ net down movements on or before $n$ and reaching $z$ at time step $n$ equals the number of paths with $m$ net down movements on or before $n$ and reaching $2m - z$ at time step $n$.

But, $z < m \iff 2m - z > m$, and from the result derived in the first case considered above,
the number of paths is
\[
\binom{n}{\frac{1}{2}(n + 2m - z)} = \binom{n}{m + j}
\]
The following price formula then applies
\[
e^{-rT} \sum_{j=0}^{n-m} C_j^n p^j (1 - p)^{n-j} H \left( S_0 u^j d^{n-j} \right)
\]
with \(C_j^n\) replaced by
\[
\binom{n}{j} \quad \text{for } z \geq m \Leftrightarrow j \leq \frac{1}{2}(n - m)
\]
\[
\binom{n}{m+j} \quad \text{for } z < m \Leftrightarrow j > \frac{1}{2}(n - m)
\]
and the limits of summation follow from noting that \(-n \leq z \leq n\) implies \(0 \leq j \leq n\), but for \(j > n - m\), no paths will touch or cross the barrier.

An up-and-in option may be analysed analogously. Let \(z^* = 2j - n\) be the net number of up movements corresponding to \(j\) up movements at time \(n\), and define
\[
m^* \equiv \left\lfloor \frac{\log \left( \frac{B}{S_0} \right)}{\log u} \right\rfloor = \frac{\log \left( \frac{B}{S_0} \right)}{\log u} + \varepsilon \quad \varepsilon \in [0, 1) \text{ constant}
\]
to be the net number of up movements required for a path to touch or exceed the barrier \(B > S_0\).

If \(z^* \geq m^*\), then the net number of up movements on or before time step \(n\) exceeds \(m^*\) and every path crosses the barrier. Hence the number of paths terminating at \(S_0 u^j d^{n-j}\) is \(\binom{n}{j}\).

If \(z^* < m^*\) then, applying the reflection principle, the number of paths with \(z^*\) net up movements that touch or cross the barrier, \(B\), on or before time step \(n\) is equal to the number of paths with \(2m^* - z^*\) net up movements,
\[
\binom{n}{\frac{1}{2}(n + 2m^* - z^*)} = \binom{n}{j - m^*}
\]
Noting that \(-n \leq z^* \leq n\) implies \(0 \leq j \leq n\), but that \(j\) must be bounded below by \(m^*\),
the price formula is
\[
e^{-rT} \sum_{j=m^*}^{n} C_j^n p^j (1 - p)^{n-j} H \left( S_0 u^j d^{n-j} \right)
\]
\(^2\)If \(j > n - m\) then the total number of down movements is less than \(m\), and so the net number of down movements is always strictly less than \(m\).
\(^3\)If \(j < m^*\) then the total number of up movements does not exceed \(m^*\), so the net number of up movements does not exceed \(m^*\) and the paths with \(j\) up movements will therefore not touch or cross the barrier \(B\).
with $\mathcal{C}_j^n$ replaced by

\[
\binom{n}{j} \quad \text{for } z^* \geq m^* \leftrightarrow j \geq \frac{1}{2}(n + m^*)
\]

\[
\binom{n}{j-m^*} \quad \text{for } z^* < m^* \leftrightarrow j < \frac{1}{2}(n + m^*)
\]

The formulae for down-and-out and up-and-out options may be recovered using the ‘in-out’ parity relationships (detailed in the Appendix).

4 Vanilla Barrier Options

4.1 Discrete-Time Formulae

A *down-and-in-call* option, maturity $T$ and strike $K$, pays $H(S_T) = \max(S_T - K, 0)$ at $T$ provided $S_t$ touches or crosses the barrier $B$ (with $S_0 > B$) on or before $T$. If $z$ denotes the number of net down movements at time step $n$, then for the path to terminate in-the-money, $S_0d^z \geq K$, so

\[
z \leq \log\left(\frac{K}{S_0}\right)
\]

Set $x = \left\lfloor \log\left(\frac{K}{S_0}\right) \right\rfloor - \gamma$ for some $\gamma \in [0, 1)$. Noting that $-n \leq z \leq x$, leads directly to $\frac{1}{2}(n - x) \leq j \leq n$. But for $j > n - m$, no paths will touch or cross the barrier. So, $j$ lies in the region $\frac{1}{2}(n - x) \leq j \leq n - m$. Moreover, $\frac{1}{2}(n - x)$ might not be an integer, and so the condition $j \geq \frac{1}{2}(n - x)$ should be replaced by $j \geq \left\lceil \frac{1}{2}(n - x) \right\rceil = \frac{1}{2}(n - x) + \delta$ for some $\delta \in [0, 1)$.

Depending on the order of $x$ and $m$ (which, given $K$ and $B$, is known) $z$ may lie either above or below $m$. If $x \leq m$, then $z \leq m$, but if $x > m$ then $z$ may satisfy $m < z \leq x$, and the range for $j$ must be split to accommodate the different path counts for $z$ below and above $m$:

- if $z > m$, then $j < \frac{1}{2}(n - m)$; so, the appropriate ranges for $j$ are $\left\lceil \frac{1}{2}(n - x) \right\rceil \leq j \leq \left\lfloor \frac{1}{2}(n - m) \right\rfloor$ and $\left\lceil \frac{1}{2}(n - m) \right\rceil \leq j \leq n - m$.

The discrete-time formulae for a *down-and-in call* are then:

\[
e^{-rT} \sum_{j=\left\lceil \frac{1}{2}(n-x) \right\rceil}^{n-m} \binom{n}{m+j} p^j (1 - p)^{n-j} (S_0d^j)^{n-j} - K
\]
If $x > m$ and $(n - m)$ is odd

$$
e^{-rT} \sum_{j=\left\lfloor \frac{1}{2}(n-m) \right\rfloor}^{\frac{n-m}{2}} \binom{n}{j} p^j (1 - p)^{n-j} (S_0 u^j d^{n-j} - K)$$

$$+ e^{-rT} \sum_{j=\left\lfloor \frac{1}{2}(n-m) \right\rfloor}^{n-m} \binom{n}{m+j} p^j (1 - p)^{n-j} (S_0 u^j d^{n-j} - K)$$

(2)

If $x > m$ and $(n - m)$ is even

$$
e^{-rT} \sum_{j=\left\lfloor \frac{1}{2}(n-m) \right\rfloor}^{\frac{1}{2}(n-m)-1} \binom{n}{j} p^j (1 - p)^{n-j} (S_0 u^j d^{n-j} - K)$$

$$+ e^{-rT} \left( \frac{n}{2} (n-m) \right) p^{\frac{1}{2}(n-m)} (1 - p)^{\frac{1}{2}(n+m)} (S_0 u^{\frac{1}{2}(n-m)} d^{\frac{1}{2}(n+m)} - K)$$

$$+ e^{-rT} \sum_{j=\left\lfloor \frac{1}{2}(n-m) \right\rfloor}^{n-m} \binom{n}{m+j} p^j (1 - p)^{n-j} (S_0 u^j d^{n-j} - K)$$

(3)

Discrete-time formulae for the remaining ‘knock-in’-options are provided in the Appendix.

4.2 Convergence to the Analytical Solution

The CRR framework of Cox, Ross and Rubenstein (1979) is a discretisation of a continuous-time environment characterised by complete markets and no-arbitrage opportunities with numeraire asset the money market account (under constant risk-free rate) and underlying stock price described by geometric Brownian motion. Continuous-time prices may be determined in this environment through an application of equivalent martingale pricing theory as described in Rubenstein and Reiner (1991). This section demonstrates that these formulae are also the limit as $n \to \infty$ of the discrete-time formulae of the previous section.

The proof of convergence follows from the application of three lemmas.

Lemma 4.1. As $n \to \infty$, $u^{-m} d^m \to \left( \frac{B}{S_0} \right)^2$.

Proof.

$$u^{-m} d^m = d^{2m} = \exp(2m \log d) = \exp \left( 2 \left( \frac{\log \left( \frac{B}{S_0} \right)}{\log d} + \varepsilon \right) \log d \right) = \left( \frac{B}{S_0} \right)^2 \exp \left( -\sigma \varepsilon \sqrt{\frac{T}{n}} \right)$$

which tends to $\left( \frac{B}{S_0} \right)^2$ as $n \to \infty$. \qed
Lemma 4.2. As \( n \to \infty \), \( (\frac{1-p}{p})^m \to \left( \frac{B}{S_0} \right)^{\frac{2r}{\sigma^2} - 1} \).

Proof.

\[
\left( \frac{1-p}{p} \right)^m = \exp \left( m \log \left( \frac{1-p}{p} \right) \right) = \exp \left( \frac{\log \left( \frac{B}{S_0} \right)}{-\sigma \sqrt{\frac{T}{n}}} + \epsilon \right) \log \left( 1 - \frac{2(r - \frac{1}{2}\sigma^2)\frac{T}{n}}{(r - \frac{1}{2}\sigma^2)\frac{T}{n} + \sigma \sqrt{\frac{T}{n}}} \right)
\]

Replacing the exponentials in the argument of the logarithm with their Maclaurin series up to order 1 in \( \frac{T}{n} \),

\[
\left( \frac{1-p}{p} \right)^m \approx \exp \left( \frac{\log \left( \frac{B}{S_0} \right)}{-\sigma \sqrt{\frac{T}{n}}} + \epsilon \right) \log \left( 1 - \frac{2(r - \frac{1}{2}\sigma^2)\frac{T}{n}}{(r - \frac{1}{2}\sigma^2)\frac{T}{n} + \sigma \sqrt{\frac{T}{n}}} \right)
\]

Then, approximating the logarithm with its Maclaurin series up to order 1

\[
\left( \frac{1-p}{p} \right)^m \approx \exp \left( \frac{\log \left( \frac{B}{S_0} \right)}{-\sigma \sqrt{\frac{T}{n}}} + \epsilon \right) \left( - \frac{2(r - \frac{1}{2}\sigma^2)\frac{T}{n}}{(r - \frac{1}{2}\sigma^2)\frac{T}{n} + \sigma \sqrt{\frac{T}{n}}} \right)
\]

\[
= \exp \left( \log \left( \frac{B}{S_0} \right) \right) \left( \frac{2(r - \frac{1}{2}\sigma^2)\frac{T}{n}}{(r - \frac{1}{2}\sigma^2)\frac{T}{n} + \sigma \sqrt{\frac{T}{n}}} \right) + \epsilon \left( - \frac{2(r - \frac{1}{2}\sigma^2)\frac{T}{n}}{(r - \frac{1}{2}\sigma^2)\frac{T}{n} + \sigma \sqrt{\frac{T}{n}}} \right)
\]

\[
= \exp \left( \log \left( \frac{B}{S_0} \right) \right) \left( \frac{2(r - \frac{1}{2}\sigma^2)\frac{T}{n}}{(r - \frac{1}{2}\sigma^2)\frac{T}{n} + \sigma \sqrt{\frac{T}{n}}} \right) - 2\epsilon(r - \frac{1}{2}\sigma^2)\frac{\sqrt{T}}{n} \frac{1}{(r - \frac{1}{2}\sigma^2)\cdot \frac{T}{n} + \sigma \sqrt{\frac{T}{n}}}
\]

which, as \( n \to \infty \), tends to

\[
\exp \left( \log \left( \frac{B}{S_0} \right) \left( \frac{2r - \frac{\sigma^2}{\sigma^2}}{\sigma^2} \right) \right) = \left( \frac{B}{S_0} \right)^{\frac{2r}{\sigma^2} - 1}
\]

\[\square\]

Lemma 4.3. Suppose that \( a = \frac{1}{2}n + \frac{\beta}{\sigma \sqrt{\frac{T}{n}}} + \rho \) for \( \beta, \rho \in \mathbb{R} \) constants. Then, as \( n \to \infty \)

\[
\sum_{i=a}^{n} \binom{n}{i} \left( p^* \right)^i (1 - p^*)^{n-i} \to \begin{cases} 
\mathcal{N} \left( \frac{-2\beta + (r + \frac{1}{2}\sigma^2)\frac{T}{n}}{\sigma \sqrt{\frac{T}{n}}} \right) & \text{for } p^* = \frac{p\rho}{e^\beta} \\
\mathcal{N} \left( \frac{-2\beta + (r + \frac{1}{2}\sigma^2)\frac{T}{n}}{\sigma \sqrt{\frac{T}{n}}} \right) & \text{for } p^* = p 
\end{cases}
\]

Proof.

\[
\sum_{i=a}^{n} \binom{n}{i} \left( p^* \right)^i (1 - p^*)^{n-i} = \text{Prob}(I \geq a) = 1 - \text{Prob}(I \leq a - 1) \quad I \sim \text{Bin}(n, p^*)
\]
And,

\[
\text{Prob}(I \leq a - 1) = \text{Prob}\left( \frac{I - np^*}{\sqrt{np^*(1 - p^*)}} \leq \frac{a - 1 - np^*}{\sqrt{np^*(1 - p^*)}} \right)
\]

with

\[
\frac{a - 1 - np^*}{\sqrt{np^*(1 - p^*)}} = \frac{1}{2}n + \frac{\beta}{\sigma \sqrt{T/n}} + \rho - 1 - np^*
\]

\[
= \frac{\beta + \rho \sigma \sqrt{T/n} + \sigma \sqrt{T/n} (n (1/2 - p^*) - 1)}{\sqrt{\sigma^2 T p^*(1 - p^*)^2}}
\]

Now, suppose that

\[p^* = \frac{pu}{e^{rT/n}} = \frac{u}{e^{rT/n}} \left( e^{rT/n} - d \right) \approx \left( \frac{1}{2} + \frac{r - \frac{1}{2} \sigma^2 rT/n}{2 \sigma \sqrt{T/n}} \right) \left( \frac{1 + \sigma \sqrt{T/n} + \frac{1}{2} \sigma^2 T/n}{1 + rT/n} \right)
\]

where the approximation follows from the substitution of first order Maclaurin expansions for \(u\) and \(e^{rT/n}\). Then

\[
\sigma \sqrt{\frac{T}{n}} \left( n \left( \frac{1}{2} - p^* \right) - 1 \right) \approx \sigma \sqrt{\frac{T}{n}} \left( n \left( \frac{1}{2} - \left( \frac{1}{2} + \frac{1}{2} \frac{(r - \frac{1}{2} \sigma^2 rT/n)}{\sigma \sqrt{T/n}} \right) \right) \left( \frac{1 + \sigma \sqrt{T/n} + \frac{1}{2} \sigma^2 T/n}{1 + rT/n} \right) \right) - 1
\]

\[
= \frac{1}{2} \left( \frac{(r - \frac{1}{2} \sigma^2 rT/n) \sigma \sqrt{T/n} - \sigma^2 T}{1 + rT/n} \right) - \frac{1}{2} \left( r - \frac{1}{2} \sigma^2 rT \right) \left( \frac{1 + \sigma \sqrt{T/n} + \frac{1}{2} \sigma^2 T/n}{1 + rT/n} \right)
\]

\[
- \sigma \sqrt{\frac{T}{n}}
\]

\[
\rightarrow - \frac{1}{2} \sigma^2 T - \frac{1}{2} \left( r - \frac{1}{2} \sigma^2 rT \right) = - \frac{1}{2} \left( r + \frac{1}{2} \sigma^2 rT \right) \text{ as } n \rightarrow \infty
\]

and

\[
p^* (1 - p^*) = \frac{p(1 - p)}{\frac{e^{2rT/n}}{n}} \approx \frac{\left( \frac{(r - \frac{1}{2} \sigma^2 rT/n) + \sigma \sqrt{T/n} + \frac{1}{2} \sigma^2 rT/n}{2 \sigma \sqrt{T/n}} \right)}{1 + 2rT/n}
\]

\[
= \frac{\sigma^2 - (r - \frac{1}{2} \sigma^2 rT/n)^2 T/n}{4 \sigma^2 + 8 \sigma^2 rT/n}
\]

\[
\rightarrow \frac{1}{4} \text{ as } n \rightarrow \infty
\]

Therefore, as \(n \rightarrow \infty\),

\[
\frac{a - 1 - np^*}{\sqrt{np^*(1 - p^*)}} \rightarrow \frac{2 \beta - (r + \frac{1}{2} \sigma^2 T \sqrt{T})}{\sigma \sqrt{T}}
\]

Following [?], pages 252-253,

\[
\text{Prob}\left( \frac{I - np^*}{\sqrt{np^*(1 - p^*)}} \leq \frac{a - 1 - np^*}{\sqrt{np^*(1 - p^*)}} \right) \rightarrow N\left( \frac{2 \beta - (r + \frac{1}{2} \sigma^2 T)}{\sigma \sqrt{T}} \right)
\]
and hence
\[ \text{Prob}(I \geq a) \to N\left( \frac{-2\beta + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \]

If
\[ p^* = p = \left( \frac{e^{r\frac{T}{n}} - d}{u - d} \right) \approx \frac{\sigma\sqrt{T} \left( n + \frac{1}{2}\sigma^2 \right)}{2\sigma\sqrt{T}} \]

then
\[ \sigma\sqrt{\frac{T}{n}} \left( n \left( \frac{1}{2} - p^* \right) - 1 \right) \approx \frac{1}{2} \left( r - \frac{1}{2}\sigma^2 \right) T - \sigma\sqrt{\frac{T}{n}} \]
\[ \to \frac{1}{2} \left( r - \frac{1}{2}\sigma^2 \right) T \text{ as } n \to \infty \]

and
\[ p^*(1 - p^*) \approx \frac{1}{4} - \frac{\left( r - \frac{1}{2}\sigma^2 \right)^2 T}{\sigma^2} \to \frac{1}{4} \text{ as } n \to \infty \]

whence
\[ \frac{a - 1 - np^*}{\sqrt{np^*(1 - p^*)}} \to \frac{2\beta - \left( r - \frac{1}{2}\sigma^2 \right) T}{\sigma\sqrt{T}} \]

and
\[ \text{Prob}(I \geq a) \to N\left( \frac{-2\beta + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \]
\[ \text{as } n \to \infty. \]

As an illustration of the convergence of the discrete-time formulae to the continuous-time formulae, consider the down-and-in call option; convergence proofs for the other barrier options are analogous. Rubenstein and Reiner (1991) deduce the following continuous-time pricing formulae for a down-and-in call option

\[ K > B \]
\[ S_0 \left( \frac{B}{S_0} \right)^{\left( \frac{2r}{\sigma^2} + 1 \right)} \mathcal{N}(d) - Ke^{-rT} \left( \frac{B}{S_0} \right)^{\left( \frac{2r}{\sigma^2} - 1 \right)} \mathcal{N}(d - \sigma\sqrt{T}) \]
\[ \text{where} \]
\[ d = \frac{2\log \left( \frac{B}{S_0} \right) - \log \left( \frac{K}{S_0} \right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \]

\[ K \leq B \]
\[ S_0 [\mathcal{N}(d_K) - \mathcal{N}(d_B)] - Ke^{-rT} \left[ \mathcal{N}(d_K - \sigma\sqrt{T}) - \mathcal{N}(d_B - \sigma\sqrt{T}) \right] \]
\[ + S_0 \left( \frac{B}{S_0} \right)^{\left( \frac{2r}{\sigma^2} + 1 \right)} \mathcal{N}(d_B') - Ke^{-rT} \left( \frac{B}{S_0} \right)^{\left( \frac{2r}{\sigma^2} - 1 \right)} \mathcal{N}(d_B' - \sigma\sqrt{T}) \]
where
\[ d_K = \frac{\log \left( \frac{S_0}{K} \right) + (r + \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}} \quad d_B = \frac{\log \left( \frac{B}{S_0} \right) + (r - \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}} \]
\[ d_B' = \frac{\log \left( \frac{B}{S_0} \right) + (r + \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}} \]

**Theorem.** As \( n \to \infty \), equation(??) → equation(??) and equations(??,??) → equation(??).

**Proof.** The proof is given for equation(??) → equation(??); the remaining cases are similar. Equation (??) is equal to
\[
S_0 \sum_{j=a_1}^{n} \binom{n}{j} \left( \frac{pu}{e^{-rT/n}} \right)^j \left( \frac{d(1-p)}{e^{-rT/n}} \right)^{n-j} - K e^{-rT} \sum_{j=a_1}^{n} \binom{n}{j} p^j (1-p)^{n-j} \\
- S_0 \sum_{j=a_2}^{n} \binom{n}{j} \left( \frac{pu}{e^{-rT/n}} \right)^j \left( \frac{d(1-p)}{e^{-rT/n}} \right)^{n-j} - K e^{-rT} \sum_{j=a_2}^{n} \binom{n}{j} p^j (1-p)^{n-j} \\
+ S_0 u^{-m} d^m \sum_{j=a_3}^{n} \binom{n}{j} \left( \frac{pu}{e^{-rT/n}} \right)^j \left( \frac{d(1-p)}{e^{-rT/n}} \right)^{n-j} \\
- K e^{-rT} \left( \frac{1-p}{p} \right)^m \sum_{j=a_3}^{n} \binom{n}{j} p^j (1-p)^{n-j}
\]

where
\[ a_1 = \left\lfloor \frac{1}{2} (n - x) \right\rfloor = \frac{1}{2} (n - x) + \delta = \frac{1}{2} n + \frac{1}{2} \log \left( \frac{K}{S_0} \right) + \left( \frac{1}{2} \gamma + \delta \right) \quad \gamma, \delta \in [0, 1) \text{ constants} \]
\[ a_2 = \left\lfloor \frac{1}{2} (n - m) \right\rfloor = \frac{1}{2} (n - m) + \delta' = \frac{1}{2} n - \frac{1}{2} \log \left( \frac{B}{S_0} \right) + \left( \delta' - \frac{1}{2} \epsilon \right) \quad \epsilon, \delta' \in [0, 1) \text{ constants} \]
\[ a_3 = \left\lfloor \frac{1}{2} (n + m) \right\rfloor = \frac{1}{2} (n + m) + \delta'' = \frac{1}{2} n - \frac{1}{2} \log \left( \frac{B}{S_0} \right) + \left( \delta'' + \frac{1}{2} \epsilon \right) \quad \epsilon, \delta'' \in [0, 1) \text{ constants} \]

Applying the lemmas, convergence follows.

### 4.3 Some Numerical Results: The Sawtooth Convergence Problem

Consider the following example of the valuation of a vanilla down-and-in call option. Let

\[ S = 95, \quad K = 100, \quad B = 90, \quad T = 1 \text{ year}, \quad \sigma = 25\%, \quad r = 10\% \text{ per annum}. \]

For comparative purposes, we have chosen the same parameter values as used by Boyle and Lau (1994). The value of a standard European call option calculated using a binomial tree
converges rapidly to the Black Scholes solution with increasing number of time steps, \( n \). As discussed earlier, this is not the case for barrier options, either when applying Boyle and Lau’s methodology or ours.

Using our model, we obtain a sawtooth like convergence to the analytical solution with spikes that are persistently below the correct value. The sawtooth convergence problem is highlighted in almost every paper written on lattice methods for barrier options Derman et al (1995). Trinomial trees, implicit, explicit and other finite-difference methods all suffer from similar problems. If we superimpose the solution and the difference between the actual and implied barriers we get

![Graph showing sawtooth convergence](image)

Figure 2: Superposition of the difference between the true & implied barrier and corresponding option values for varying numbers of time steps. (The continuous-time solution is represented by the horizontal line at 5.6605.)

This figure demonstrates the predictable result that the option value is always closest to the analytical solution when the barrier implied by the tree is closest to the actual barrier specified in the contract. It is clear from this that the “option specification error” is responsible for the poor convergence exhibited by lattice methods in barrier option evaluation. Note also the persistent bias associated with our solution in comparison to the analytical solution.
5 Conclusion

Binomial lattices provide an intuitive and flexible approach to the discrete-time pricing of financial derivatives. However, an application of this approach to the pricing of barrier options introduces two difficulties: (i) the barrier, in general, lies between lattice nodes resulting in a mispricing of the option; and (ii) the path dependent nature of barrier options has necessitated a computationally expensive backward induction algorithm to determine the price from the lattice.

This paper addresses the second difficulty. A closed-form, discrete-time solution, analogous to that of Cox, Ross and Rubenstein (1979) and applicable to all European barrier options with constant barrier, is deduced. In the case of vanilla barrier options, this closed-form solution is shown to converge to the known continuous-time solution.

References


6 Appendix

Below are the discrete-time formulae for ‘knock-in’-options (the formula for a down-and-in call option is reproduced for ease of reference).

In these formulae

\[ x = \left\lfloor \log \left( \frac{K}{S_0} \right) \log d \right\rfloor \]
\[ m = \left\lceil \log \left( \frac{B}{S_0} \right) \log d \right\rceil \]
\[ x^* = \left\lfloor \log \left( \frac{K}{S_0} \right) \log u \right\rfloor \]
\[ m^* = \left\lceil \log \left( \frac{B}{S_0} \right) \log u \right\rceil \]

Also, define the following indices, which are always integers or half-integers:

\[ I_1 = \frac{1}{2} (n + m^*) \quad I_2 = \frac{1}{2} (n + x^*) \quad I_3 = \frac{1}{2} (n - m) \]
\[ I_4 = \frac{1}{2} (n - x) \quad I_5 = \frac{1}{2} (n + x) \]
\[ I_{i,+} = \lceil I_i \rceil \quad I_{i,-} = \lfloor I_i \rfloor \]

Up-and-In Call Option

If \( x^* > m^* \)

\[ c_{u-i} = e^{-rT} \sum_{j=I_{2,+}}^{n} \binom{n}{j} p^j (1-p)^{n-j} (S_0 u^j d^{n-j} - K) \]

If \( x^* \leq m^* \) and \( (n + m^*) \) is odd

\[ c_{u-i} = e^{-rT} \left[ \sum_{j=I_{2,+}}^{I_{1,-}} \binom{n}{j} p^j (1-p)^{n-j} (S_0 u^j d^{n-j} - K) + \sum_{j=I_{1,+}}^{n} \binom{n}{j} p^j (1-p)^{n-j} (S_0 u^j d^{n-j} - K) \right] \]

If \( x^* \leq m^* \) and \( (n + m^*) \) is even

\[ c_{u-i} = e^{-rT} \left[ \sum_{j=I_{2,+}}^{I_{1,-}-1} \binom{n}{j} p^j (1-p)^{n-j} (S_0 u^j d^{n-j} - K) + \sum_{j=I_{1,+}}^{n} \binom{n}{j} p^j (1-p)^{n-j} (S_0 u^j d^{n-j} - K) \right] \]

Up-and-In Put Option

If \( x^* \leq m^* \)

\[ p_{u-i} = e^{-rT} \sum_{j=m^*}^{I_{3,+}} \binom{n}{j} p^j (1-p)^{n-j} (K - S_0 u^j d^{n-j}) \]

If \( x^* > m^* \) and \( (n + m^*) \) is odd

\[ p_{u-i} = e^{-rT} \left[ \sum_{j=m^*}^{I_{3,+}-1} \binom{n}{j} p^j (1-p)^{n-j} (K - S_0 u^j d^{n-j}) + \sum_{j=I_{1,+}}^{I_{2,+}-1} \binom{n}{j} p^j (1-p)^{n-j} (K - S_0 u^j d^{n-j}) \right] \]
If \( x^* > m^* \) and \((n + m^*)\) is even

\[
p_{u-i} = e^{-rT} \left[ \sum_{j=m^*}^{I_1-1} \left( \frac{n}{j - m^*} \right) p^j (1 - p)^{n-j} (K - S_0 w^j d^{n-j}) + \sum_{j=1}^{I_2-1} \left( \frac{n}{j} \right) p^j (1 - p)^{n-j} (K - S_0 w^j d^{n-j}) \right]
\]

**Down-and-In Call Option**

If \( x \leq m \)

\[
c_{d-i} = e^{-rT} \sum_{j=\left[\frac{1}{2}(n-x)\right]}^{n-m} \left( \frac{n}{m + j} \right) p^j (1 - p)^{n-j} (S_0 w^j d^{n-j} - K)
\]

If \( x > m \) and \((n - m)\) is odd

\[
c_{d-i} = e^{-rT} \left[ \sum_{j=I_3 + 1}^{I_2-1} \left( \frac{n}{j} \right) p^j (1 - p)^{n-j} (S_0 w^j d^{n-j} - K) + \sum_{j=I_3}^{n-m} \left( \frac{n}{m + j} \right) p^j (1 - p)^{n-j} (S_0 w^j d^{n-j} - K) \right]
\]

If \( x > m \) and \((n - m)\) is even

\[
c_{d-i} = e^{-rT} \left[ \sum_{j=I_3 + 1}^{I_2-1} \left( \frac{n}{j} \right) p^j (1 - p)^{n-j} (S_0 w^j d^{n-j} - K) + \sum_{j=I_3}^{n-m} \left( \frac{n}{m + j} \right) p^j (1 - p)^{n-j} (S_0 w^j d^{n-j} - K) \right]
\]

**Down-and-In Put Option**

If \( x > m \)

\[
p_{d-i} = e^{-rT} \sum_{j=0}^{I_5-1} \left( \frac{n}{j} \right) p^j (1 - p)^{n-j} (K - S_0 w^j d^{n-j})
\]

If \( x \leq m \) and \((n - m)\) is odd

\[
p_{d-i} = e^{-rT} \left[ \sum_{j=0}^{I_5-1} \left( \frac{n}{j} \right) p^j (1 - p)^{n-j} (K - S_0 w^j d^{n-j}) + \sum_{j=I_3}^{I_5} \left( \frac{n}{m + j} \right) p^j (1 - p)^{n-j} (K - S_0 w^j d^{n-j}) \right]
\]

If \( x \leq m \) and \((n - m)\) is even

\[
p_{d-i} = e^{-rT} \left[ \sum_{j=0}^{I_5-1} \left( \frac{n}{j} \right) p^j (1 - p)^{n-j} (K - S_0 w^j d^{n-j}) + \sum_{j=I_3}^{I_5} \left( \frac{n}{m + j} \right) p^j (1 - p)^{n-j} (K - S_0 w^j d^{n-j}) \right]
\]

**Out Options**

The discrete-time prices of the ‘knock-out’-options may then be determined from ‘in-out’ parity:

\[
\text{price of a vanilla call} = \text{price of an up-and-in call} + \text{price of an up-and-out call} = \text{price of a down-and-in call} + \text{price of a down-and-out call}
\]

\[
\text{price of a vanilla put} = \text{price of an up-and-in put} + \text{price of an up-and-out put} = \text{price of a down-and-in put} + \text{price of a down-and-out put}
\]