Behavioral Expectations in Nonlinear DSGE Models

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Motivation

- Cognitive discounting (CD), a tractable approach to incorporating bounded rationality (BR) pioneered by (**Gabaix 2020**), enriches New Keynesian models by
  - Introducing **Behavioral realism**: agents may not fully comprehend events far into the future.
  - Enhancing **model sophistication**: enabling nuanced analysis of monetary and fiscal policies, addressing issues such as the FG puzzle and equilibrium determinacy.
  - Aligning with **empirical observations:** improving model fitting and empirical relevance.
- CD originated within the confines of a canonical, **small**, **linearized** rational expectations model: clear examination of the impact of bounded rationality on economic outcomes and policy decisions.

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Questions

The application of CD to larger, nonlinear frameworks raises questions relative to:

- Generalizing CD to accommodate large-scale and nonlinear models.
- Extending insights from linear to nonlinear models.
- Exploring additional insights nonlinear models offer beyond those from linear models.

Connections with the literature

- Cognitive discounting (**linear**): Gabaix (2020), Hohberger et. al (2024)
- Regime-switching in DSGE models: Davig & Doh (2011) Bianchi (2013), many more... $\rightarrow$ **RE**
- ZLB: Nakata (2017 and others), Adam and Billi (2007), Fernandez-Villaverde et al. (2015), many more... $\rightarrow$ **RE**

The angle of attack

- Computations utilize the RISE toolbox (**Maih 2015**), offering:
    - Tools for nonlinear and nonstationary models
    - Support for regime switching
    - Bayesian estimation capabilities
    - Options for optimal policy analysis: commitment, discretion, loose commitment, stochastic replanning
    - Higher-order perturbation techniques
- The strategy involves defining a single behavioral expectation operator for the entire DSGE system and specifying variables subject to the behavioral scheme.

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A Nonlinear Foundation for the Specification in **Gabaix (2020)**

Gabaix (2020) - (log-)linear environment

$$E_t^{BR,linear}\left[x_{t+1}-x^*\right]=\overline{m}E_t\left[x_{t+1}-x^*\right] \quad (1)$$

Our Generalization - nonlinear environment

$$E_t^{BR}\left[\frac{X_{t+1}}{X^*}\right]=E_t\left[\left(\frac{X_{t+1}}{X^*}\right)^{\rho_X}\right] \quad (2)$$

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From Rational to Behavioral Expectations

The constant-parameter case I

Given a DSGE model

$$E_t f_\theta(X_{t+1}, X_t, X_{t-1}, \varepsilon_t) = 0 \tag{3}$$

where

- X_t is a vector of endogenous variables
- ε_t is a vector of exogenous shocks
- E_t is the rational expectations operator
- f_θ is a vector of functions
- θ is the vector of parameters

The constant-parameter case II

We introduce the E_t^{BR} operator with variable-specific cognitive discounting parameter ρ_X:

$$E_t^{BR} f_\theta(X_{t+1}, X_t, X_{t-1}, \varepsilon_t) = E_t f_{\{\theta,\rho_X\}}(X \odot (X_{t+1} \oslash X)^{\circ \rho_X}, X_t, X_{t-1}, \varepsilon_t) \quad (4)$$

where

- X is the steady state
- $0 \leq \rho_X \leq 1$ is a vector of cognitive discounting parameters specific to each variable in vector X

$$log(X \odot (X_{t+1} \oslash X)^{\circ \rho_X}) = log(X) + \rho_X \odot [log(X_{t+1}) - log(X)] \quad (5)$$

- when $\rho_X = 1$, E_t and E_t^{BR} coincide
- when $\rho_X = 0$, X_{t+1} becomes X

Is Cognitive Discounting Difficult to Implement in RISE?

NO!

```
@cognitive_discount C, Y, ...
```

Differences and similarities with Gabaix

- Similarities
  - Tilting of forward-looking terms towards their steady state
  - Up to first order, our expression for E_t^{BR} coincides with Gabaix
- Differences
  - We define BE on the system of equations and select the variables entering the scheme. So we apply a variable-specific discounting while Gabaix applies agent-specific discounting
  - If a variable is selected for BE, it is treated the same way everywhere it appears in the system.
  - We can handle nonlinear and nonstationary models

Adding Regime switching

Adding regime switching is straightforward. The previous problem becomes

$$
\begin{aligned}
0 &= E_t^{BR} \sum_{j=1}^{h} \pi_{r_t,j}(X_t(r_t)) f(X_{t+1}(j), X_t(r_t), X_{t-1}, \varepsilon_t, \theta(r_t)) \\
&= E_t \sum_{j=1}^{h} \pi_{r_t,j}(X_t(r_t)) f(X(j) \odot (X_{t+1}(j) \oslash X(j))^{\circ \rho_X(j)}, X_t(r_t), X_{t-1}, \varepsilon_t, \theta(r_t))
\end{aligned} \tag{6}
$$

where

- $\pi_{r_t,j}(X_t(r_t))$ is the (possibly time-varying) transition probability of going from regime r_t today to regime j tomorrow
- $\rho_X(j)$ is the vector of variable-specific discounting parameters
- We can approximate around regime-dependent reference points $X(j)$

An Endowment Economy

A Nonlinear Model

Consider the following quadratic model for log consumption $c_t \equiv log(C_t)$ inspired by Aruoba et al (2017):

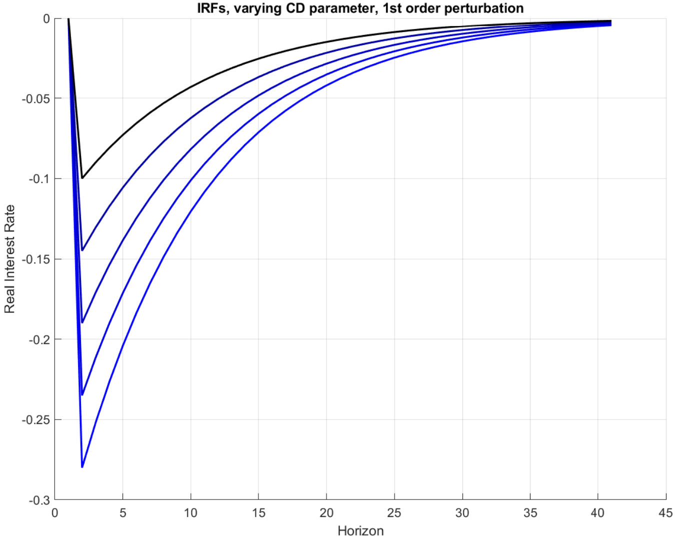
$$c_t = m_c + \phi_1 c_{t-1} + \phi_2 s_{t-1}^2 + (1 + \gamma s_{t-1})\sigma_1 u_t + \sigma_2 u_t^2 \quad (7)$$

$$s_t = \phi_1 s_{t-1} + \sqrt{1 - \phi_1^2} u_t, \, u_t \sim N(0, 1) \quad (8)$$

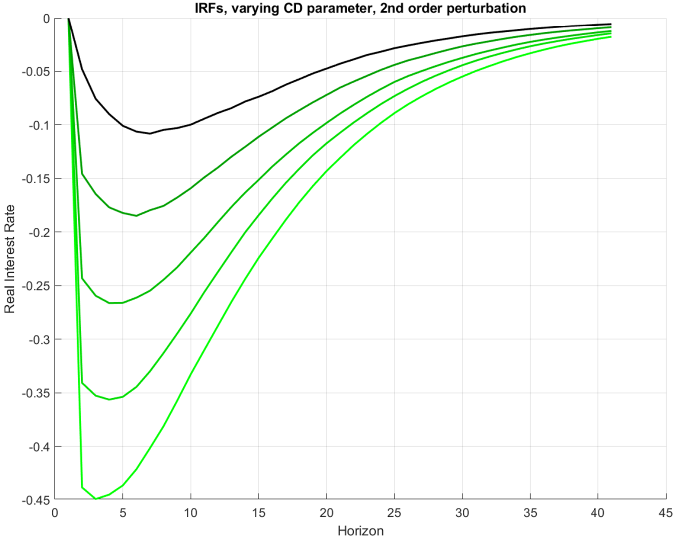
- To assess the impact of nonlinearities and cognitive discounting, we add a consumption Euler equation for a one-period riskless bond with real gross return R_t (assuming log utility):

$$\frac{1}{C_t} = \beta R_t E_t^{BR} \left(\frac{1}{C_{t+1}} \right)$$

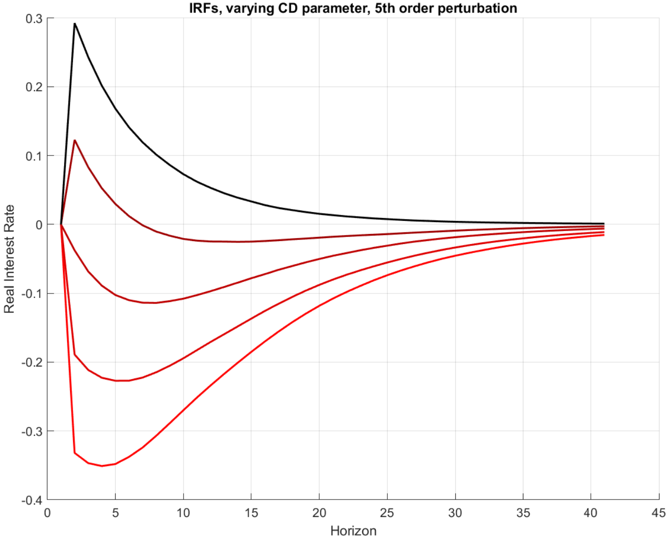
First-order perturbation



Second-order perturbation



Fifth-order perturbation



A New Keynesian Model

A simple NK DSGE model: Nakata(2017) I

Extended to include monetary policy, productivity and cost-push shocks, trend inflation, and a smooth nonlinearity in the Taylor rule. The equilibrium conditions are given by:

$$(\delta_t - 1) = \rho_\delta (\delta_{t-1} - 1) + \sigma_\delta \varepsilon_{\delta,t} \qquad \text{(Preference shock)}$$

$$\log (A_t) = \rho_A \log (A_{t-1}) + \sigma_A \varepsilon_{A,t} \qquad \text{(Technology shock)}$$

$$\log \left(\frac{\theta_t}{\theta} \right) = \rho_\theta \log \left(\frac{\theta_{t-1}}{\theta} \right) - \sigma_\theta \varepsilon_{\theta,t} \qquad \text{(Cost-push shock)}$$

A simple NK DSGE model: Nakata(2017) II

$$\frac{N_t}{C_t^{\chi_c}}\left[\varphi\left(\Pi_t - \Pi_{ss}\right)\Pi_t - (1-\theta_t) - \theta_t w_t\right] = \beta\delta_t E_t \frac{N_{t+1}}{C_{t+1}{}^{\chi_c}}\varphi\left(\Pi_{t+1} - \Pi_{ss}\right)\Pi_{t+1} \tag{9}$$

$$Y_t = C_t + \frac{\varphi}{2}\left[\Pi_t - \Pi_{ss}\right]^2 Y_t \tag{10}$$

$$Y_t = A_t N_t \tag{11}$$

$$\frac{R_t}{R_{ss}} = E_t\left(\frac{R_{t-1}}{R_{ss}}\right)^{\phi_r}\left[\left(\frac{\Pi_{t+1}}{\Pi_{ss}}\right)^{\phi_\pi}\exp\left(\phi_{\pi,a}\frac{\Pi_t}{\Pi_{ss}}\right)\right]^{(1-\phi_r)}\exp(\sigma_R\varepsilon_{R,t}) \tag{12}$$

$$C_t^{-\chi_c} = \beta E_t \delta_t R_t C_{t+1}{}^{-\chi_c}\Pi_{t+1}{}^{-1} \tag{13}$$

$$w_t = N_t^{\chi_n} C_t^{\chi_c} \tag{14}$$

A simple NK DSGE model: Nakata(2017) III

What can go wrong

- How to handle multiple forward-looking terms in the same equation? e.g. C_{t+1}, N_{t+1}, and Π_{t+1}
- How to deal with situations where some forward-looking terms appear in multiple equations? e.g. C_{t+1} and Π_{t+1}
- The basic CD framework does not handle these cases

Model parameterization & variations I

Parameterization

- Parameters from Nakata
- cognitive discount set to 0.85
- increased standard deviations to make the model more nonlinear

Model variants

- Rational Expectations: RE
- Cognitive discounting on inflation: CD(PAI)
- Cognitive discounting on consumption and inflation: CD(C,PAI)

Model parameterization & variations II

Table: Parameter Values

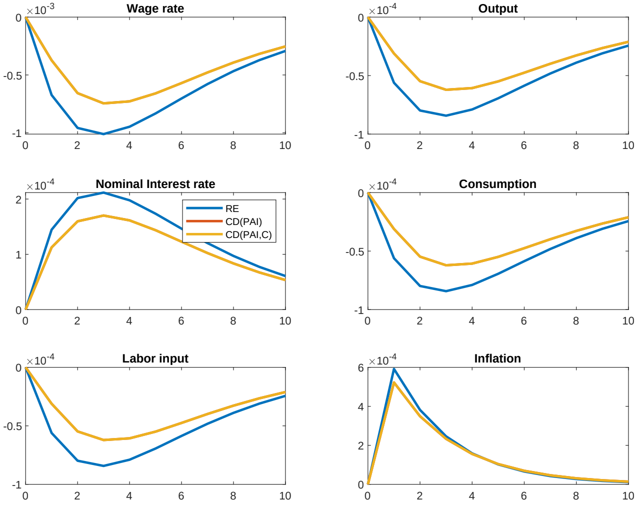
Parameter	Value	Parameter	Value
β	0.9907	ρ_θ	0.75
χ_c	10	σ_θ	0.05
χ_n	2	ρ_a	0.75
θ	11	σ_a	0.002
φ	200	σ_r	0.007
ϕ_π	1.5	$\phi_{\pi a}$	0.0075
ρ_δ	0.8	ϕ_r	0.75
σ_δ	0.0029	π	$(1+\frac{2.5}{100})^{0.25}$
cogn.d	**0.85**		

Linear adventures: Taylor principle easier to satisfy

ϕ_π	RE	CD(PAI)	CD(C,PAI)
3	1	1	1
2	1	1	1
1	0	0	1
0.9	0	0	1

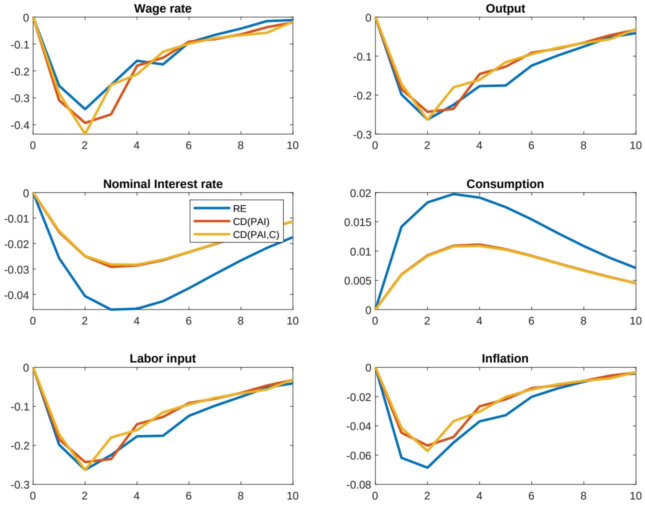
Table: ϕ_π: interest reaction to inflation

Cost-push shock



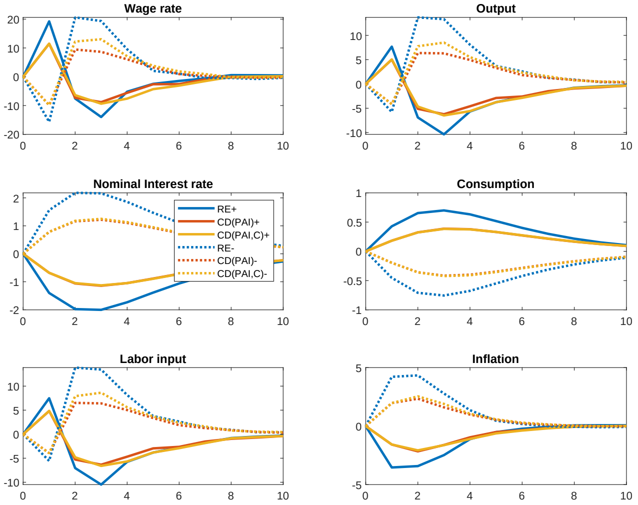
- First-order approximation

Cost-push shock



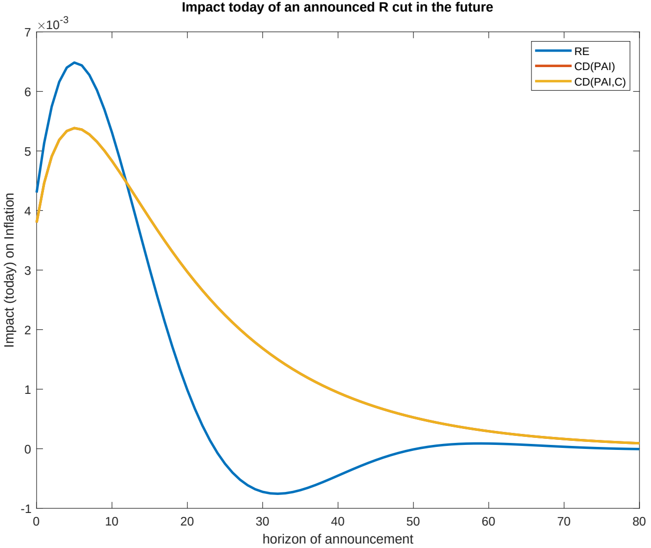
- Fifth-order approximation: qualitative changes (consumption, interest rate, inflation) + CD no necessarily less volatile than RE

Asymmetries: Productivity shock



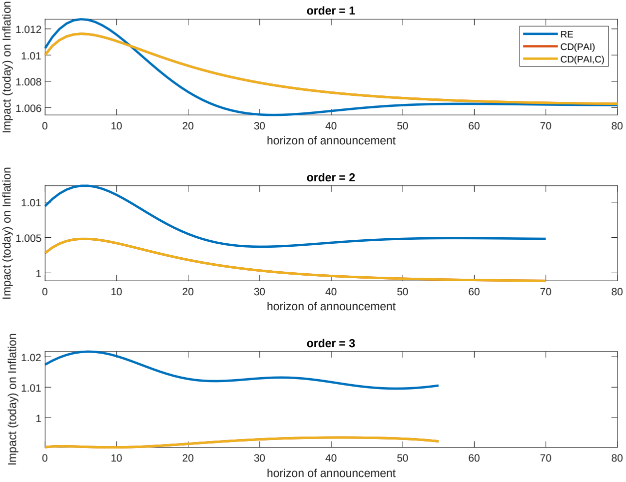
- Fifth-order approximation: Wage & Labor & output & inflation

Forward guidance : $x_t = T\left(x_{t-1}, \varepsilon_{t|t}, \varepsilon_{t+1|t}, \ldots, \varepsilon_{t+k|t}\right)$



- no puzzle + CD less volatile than RE

FG: Impact today of an announced R cut in the future



- no puzzle + CD less volatile than RE
- Differences between CD and RE increase with the order of approximation

Switches in Bounded Rationality

What Happens to Expectations at the ZLB?

Going beyond

- A simple model is useful for illustrating the core implications and mechanisms
- A larger model is necessary to confront the data

Agenda

- We consider a DSGE model where switches in parameters lead the economy to the ZLB
- We also allow the cognitive discounting parameter to switch at those times
- What is the impact of those changes in cognitive discounting?

Our Laboratory: Belongia & Ireland (2022)

- habit formation
- incomplete indexation
- Taylor rule

Belongia and Ireland focus on comparing the performance of an estimated Taylor rule to the counterfactual of money growth rules.

Using the same Taylor-rule model, We focus on modeling the ZLB using regime-switching in the presence of behavioral expectations, allowing for switching cognitive discounting parameters.

Markov Switching

- Markov switching in policy rule parameters, cognitive discounting parameters *governed by same Markov chain*

$$\frac{R_t}{R_{ss}(r_t)} = \left\{ \left(\frac{R_{t-1}}{R_{ss}(r_t)} \right)^{\rho_R} \left[\left(\frac{\pi_t}{\pi_{ss}} \right)^{\phi_{R,\pi}} \left(\frac{y_t}{y_{ss}} \right)^{\phi_{R,y}} \right]^{1-\rho} \right\}^{n(r_t)} \exp(\sigma_R(r_t)\varepsilon_{R,t}) \tag{15}$$

where $n(r_t)$ and $\sigma_R(r_t)$ are switching parameters

- In normal times, $n = 1$ and the interest rate follows a Taylor rule

- At the ZLB, the interest rate is equal to its regime-dependent mean plus a small perturbation

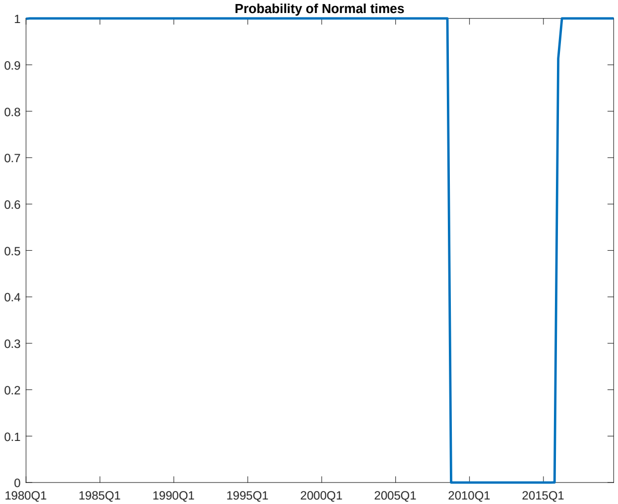
- r_t is a Markov process in two states

Estimation results: ZLB + CD on all but the exogenous processes: Sample 1980Q1 : 2019Q4

Parameter	Distribution	Prior Mean	Prior Std	Mode
pol_tp_1_2	BETA	0.0500	0.1000	0.00081974
pol_tp_2_1	BETA	0.1000	0.1000	0.022035
mcd_C_pol_1	UNIFORM	0.5000	0.0833	5.0319e-11
mcd_C_pol_2	UNIFORM	0.5000	0.0833	1
mcd_M_pol_1	UNIFORM	0.5000	0.0833	0.94118
mcd_M_pol_2	UNIFORM	0.5000	0.0833	0.87913
mcd_PAI_pol_1	UNIFORM	0.5000	0.0833	5.2779e-11
mcd_PAI_pol_2	UNIFORM	0.5000	0.0833	0.16476
mcd_Q_pol_1	UNIFORM	0.5000	0.0833	1
mcd_Q_pol_2	UNIFORM	0.5000	0.0833	0.46448
mcd_Y_pol_1	UNIFORM	0.5000	0.0833	0.97902
mcd_Y_pol_2	UNIFORM	0.5000	0.0833	0.98499
mcd_LAMBDA_pol_1	UNIFORM	0.5000	0.0833	5.2691e-11
mcd_LAMBDA_pol_2	UNIFORM	0.5000	0.0833	0.96832

- state 1: normal times
- state 2: ZLB times
- C: consumption, M: Money, PAI: Inflation, Q: Efficient level of output, Y: Output, LAMBDA: Lagrange Mult. BC

Probablity of Normal Times



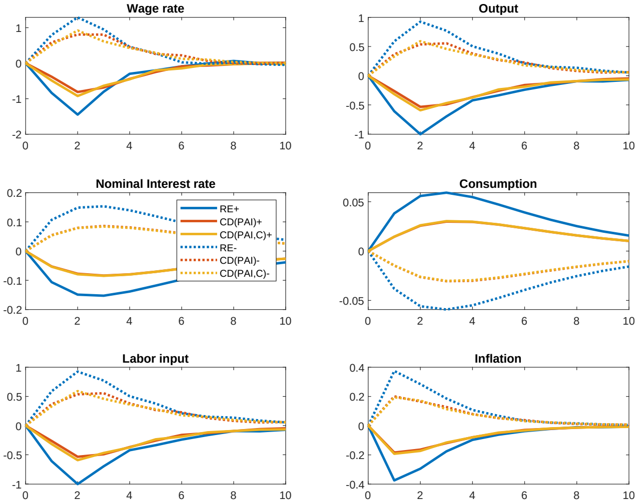
Conclusion: The story so far

- New approach to cognitive discounting: Linear, nonlinear, nonstationary, regime-switching models.
- Linear model results may not generalize to nonlinear models.
- Forward-guidance less powerful - no forward guidance puzzle!
- ZLB can be more costly depending on how CD is specified
- Optimal policy mitigates differences between rational expectations (RE) and cognitive discounting (CD)
- Higher-order approximations reveal meaningful asymmetries.
- The effects of cognitive discounting vary based on the variable and on the regime

Thank you!

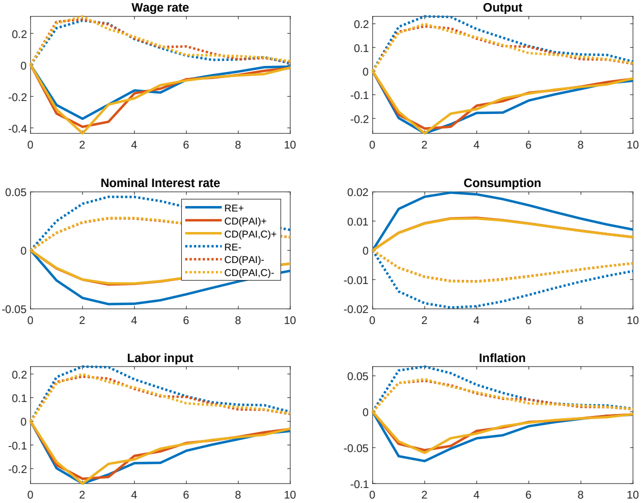
Appendices to the Nakata Model

Asymmetries: Discount factor shock



- Fifth-order approximation: Wage & Labor & output & inflation

Asymmetries: Cost-push shock



- Fifth-order approximation: Wage & Labor & output & inflation

Taylor-rule model: Moments

	order = 1	order = 2	order = 3	order = 4	order = 5
W: std(CD(PAI))/std(RE)	1.1229	0.031139	0.079818	NaN	NaN
W: std(CD(PAI,C))/std(RE)	1.1255	0.059187	2.3621e-07	NaN	NaN
Y: std(CD(PAI))/std(RE)	0.79478	0.039961	0.18719	0	NaN
Y: std(CD(PAI,C))/std(RE)	0.78663	0.044547	0.00043577	0	NaN
R: std(CD(PAI))/std(RE)	0.84435	0.83219	0.73501	0.59874	26258.775
R: std(CD(PAI,C))/std(RE)	0.83133	0.8237	0.75759	0.34922	0.0031944
C: std(CD(PAI))/std(RE)	0.79478	0.8104	0.73333	1.2038	3.7909
C: std(CD(PAI,C))/std(RE)	0.78663	0.79864	0.75596	1.199	0.02562
PAI: std(CD(PAI))/std(RE)	0.91329	0.83245	0.73214	0.00011064	NaN
PAI: std(CD(PAI,C))/std(RE)	0.90436	0.84165	0.46271	2.0319e-10	1.1873e-17

Table: Results from 10 000 pruned simulations with 1000 burn-in

Ramsey model: Moments I

	order = 1	order = 2	order = 3	order = 4	order = 5
W: std(CD(PAI))/std(RE)	0.98112	0.96259	0.96597	0.95435	0.81244
W: std(CD(PAI,C))/std(RE)	0.98371	0.94822	0.96156	1.0049	0.82892
Y: std(CD(PAI))/std(RE)	1.0162	1.0182	1.0234	1.0128	0.97618
Y: std(CD(PAI,C))/std(RE)	0.99882	0.98992	1.0145	1.02	0.97586
R: std(CD(PAI))/std(RE)	1.0225	1.0235	1.0207	1.0065	0.91922
R: std(CD(PAI,C))/std(RE)	1.0026	0.99698	1.01	1.0319	0.95739
C: std(CD(PAI))/std(RE)	1.0162	1.0183	1.0234	1.0123	0.97606
C: std(CD(PAI,C))/std(RE)	0.99882	0.99	1.0144	1.0197	0.97608
N: std(CD(PAI))/std(RE)	1.0339	1.0333	1.0336	1.0236	1.0134
N: std(CD(PAI,C))/std(RE)	1.004	0.99977	1.015	1.0071	1.0079
PAI: std(CD(PAI))/std(RE)	1.0045	1.0042	1.0416	1.0381	0.94718
PAI: std(CD(PAI,C))/std(RE)	1.0104	0.99656	1.041	1.0682	1.0133

Table: Results from 10 000 pruned simulations with 1000 burn-in

Ramsey model: Moments II

- **Varying Effects with Order of Approximation:** The level of approximation in the DSGE model affects the relative volatility of variables under different expectation schemes.
- **Convergence of CD(PAI) to RE:** Y and C, the ratio of standard deviations between CD(PAI) and RE tends to hover around 1 or slightly above across different orders of approximation. This suggests that CD(PAI) converges to rational expectations (RE) for these variables, especially at higher orders of approximation.
- **Divergence with CD(PAI,C):** In some cases, such as inflation (PAI), the standard deviations under CD(PAI,C) compared to RE are consistently higher across different orders of approximation. This indicates that incorporating cognitive factors into expectations for multiple variables can lead to higher volatility compared to rational expectations.
- **Non-monotonic Patterns:** e.g. labor (N) under CD(PAI) and CD(PAI,C).

Discretion model: Moments I

	order = 1	order = 2	order = 3	order = 4	order = 5
W: std(CD(PAI))/std(RE)	1.0001	0.97701	1.0114	1.0114	1.0042
W: std(CD(PAI,C))/std(RE)	0.99432	0.98119	0.98553	0.98202	1.0379
Y: std(CD(PAI))/std(RE)	1.006	0.98957	1.0018	0.98037	1.0103
Y: std(CD(PAI,C))/std(RE)	1.0114	1.0009	1.0025	1.0076	1.0104
R: std(CD(PAI))/std(RE)	1.0123	0.99574	1.014	0.98029	1.0214
R: std(CD(PAI,C))/std(RE)	1.0145	0.9996	1.0175	1.0004	1.0097
C: std(CD(PAI))/std(RE)	1.006	0.98957	1.0018	0.98037	1.0103
C: std(CD(PAI,C))/std(RE)	1.0114	1.0009	1.0025	1.0076	1.0104
N: std(CD(PAI))/std(RE)	1.0138	0.99169	1.0212	0.97354	1.0273
N: std(CD(PAI,C))/std(RE)	1.0135	0.99566	1.026	0.99253	1.0058
PAI: std(CD(PAI))/std(RE)	1.0075	0.99086	0.98945	0.97821	1.0077
PAI: std(CD(PAI,C))/std(RE)	1.0114	1.0071	0.99195	1.0128	1.0145

Table: Results from 10 000 pruned simulations with 1000 burn-in

Discretion model: Moments II

Similarities to Ramsey

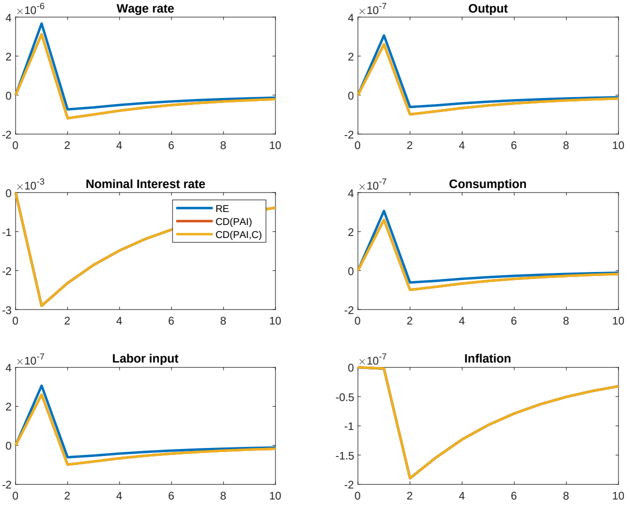
- **Variable Sensitivity:** variables like output (Y) and consumption (C) tend to show relatively smaller deviations from rational expectations across different orders of approximation.
- **Non-monotonic Patterns:**

Differences

- **Impact of CD(PAI,C):** Under Ramsey, this scenario often leads to higher volatility compared to rational expectations, especially for inflation (PAI). Under discretion, while there are instances where CD(PAI,C) leads to higher volatility, in some cases, it results in volatility levels closer to rational expectations or even lower, especially at higher orders of approximation.
- **Convergence of CD(PAI):** Under Ramsey, CD(PAI) tends to converge to rational expectations for variables like output (Y) and consumption (C) as the order of approximation increases. Under discretion, the convergence is less consistent, with some deviations persisting even at higher orders of approximation.

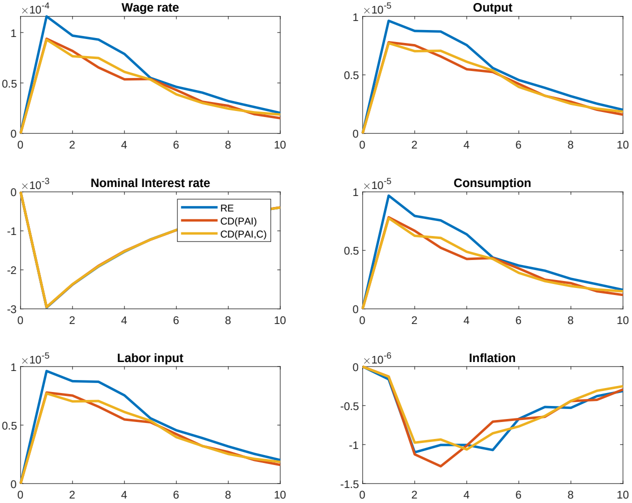
Nakata model: Optimal Policy: Further shocks

Ramsey Policy: Discount factor shock



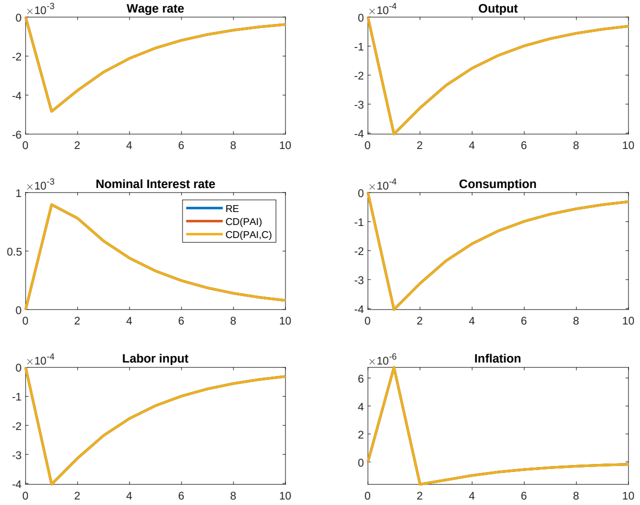
- First-order approximation: RE & CD : insignificant differences

Ramsey Policy: Discount factor shock



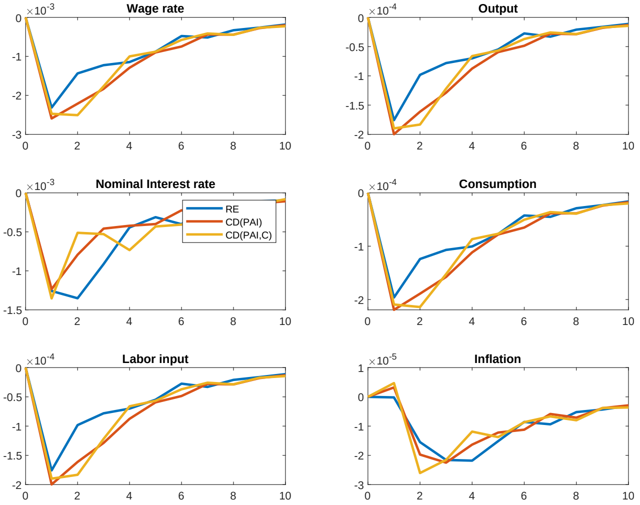
- Fifth-order approximation: RE & CD : more differences

Ramsey Policy: Cost-push shock



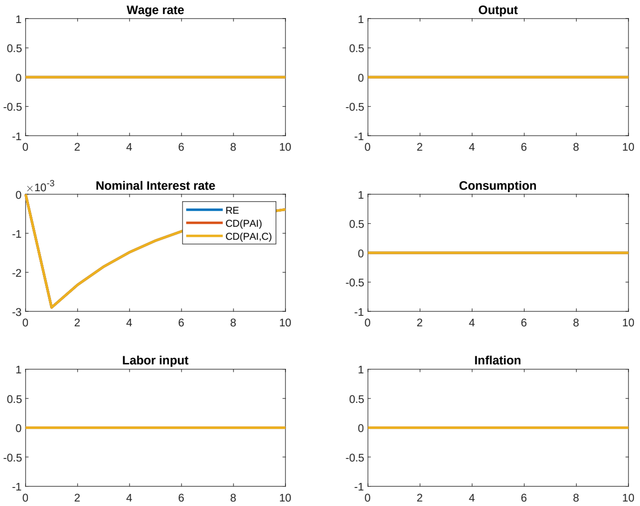
- First-order approximation: RE & CD : insignificant differences

Ramsey Policy: Cost-push shock



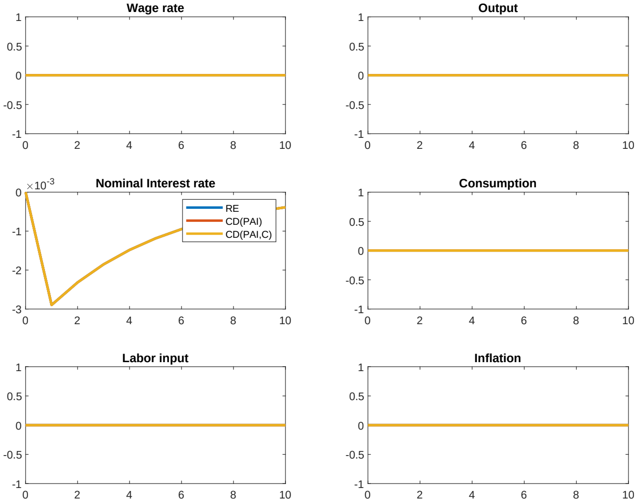
- Fifth-order approximation: RE & CD : more differences

Discretionary Policy: Discount factor shock



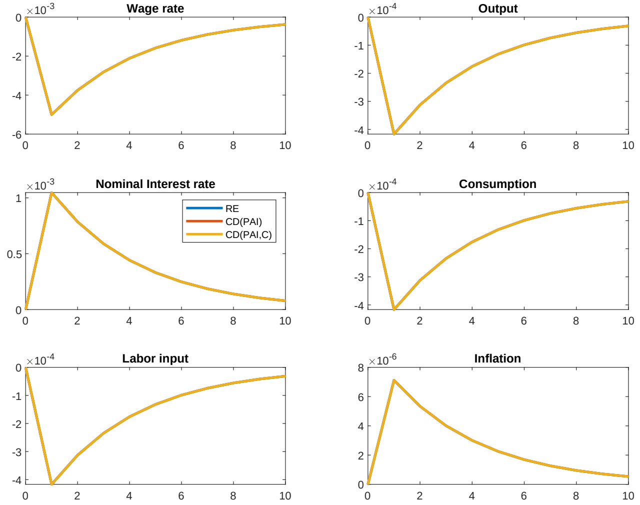
- First-order approximation: RE & CD : insignificant differences + perfect stabilization overall

Discretionary Policy: Discount factor shock



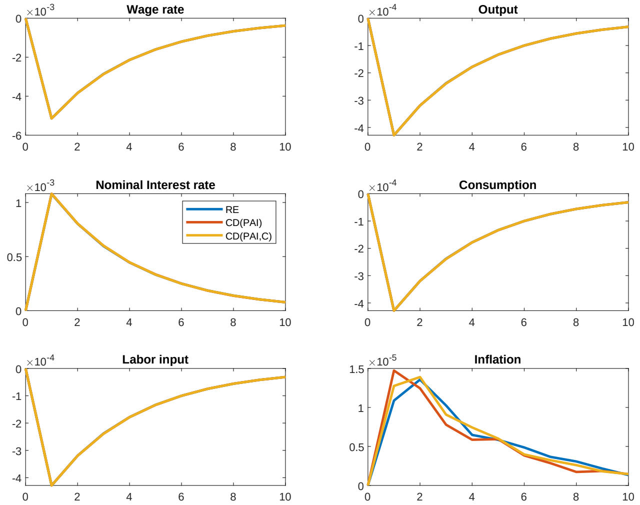
- Fifth-order approximation: RE & CD : insignificant differences + perfect stabilization overall

Discretionary Policy: Cost-push shock



- First-order approximation: RE & CD : insignificant differences

Discretionary Policy: Cost-push shock



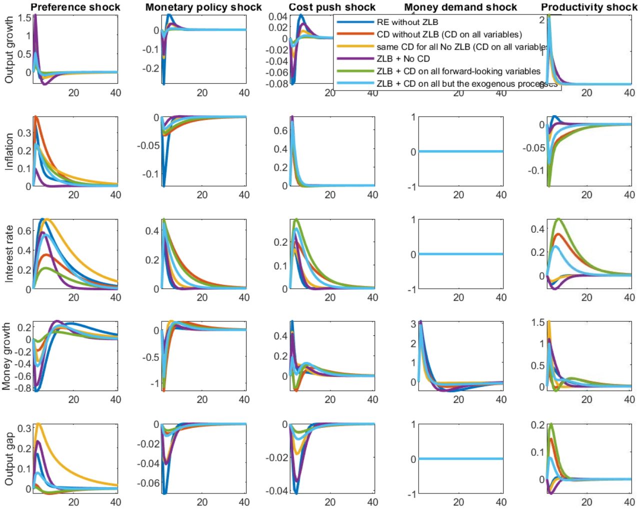
- Fifth-order approximation: RE & CD: insignificant differences, except for inflation (of course?)

Appendices to the Belongia-Ireland model

List of rival models

- no CD and no ZLB
- CD without ZLB (CD on all variables)
- same CD for all No ZLB (CD on all variables)
- ZLB + No CD
- ZLB + CD on all forward-looking variables
- ZLB + CD on all but the exogenous processes

Dynamics in normal times



Dynamics at the ZLB