A unified approach to Determinacy Conditions with Regime Switching

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Incipit

Determinacy conditions and regime switching

Context

- ◇ Conditions ensuring the existence of a unique stable equilibrium
- ◇ In Rational-expectations models with regime switching

Why

- ◇ Determinacy: Feature of a good model or of a good policy
- ◇ More complex than for linear models & depends on stability concept

What we do

- ◇ Unified framework for mean-square stability and boundedness
- ◇ Characterize the different equilibria for each stability concept
- ◇ Applications to standard models with monetary/fiscal switching

This paper: The epilogue of 17 years of research in 7 chapters

Chapter 1: The Prologue

Davig and Leeper, AER (2007): key insights

◇ New-keynesian model with monetary switching

$$\begin{aligned} x_t &= \mathbb{E}_t x_{t+1} - \sigma^{-1}\left[\alpha(s_t)\pi_t - \mathbb{E}_t \pi_{t+1}\right] + \varepsilon_t^D \\ \pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + \varepsilon_t^S \end{aligned}$$

◇ Model that can be written as

$$X_t = A(s_t)\mathbb{E}_t X_{t+1} + B(s_t)\varepsilon_t$$

◇ Solving forward leads to

$$X_t = B(s_t)\varepsilon_t + \sum_{k=1}^{p} \mathbb{E}_t A(s_t)\cdots A(s_{t+k-1})B(s_{t+k})\varepsilon_{t+k} + \mathbb{E}_t A(s_t)\cdots A(s_{t+p})X_{t+p}$$

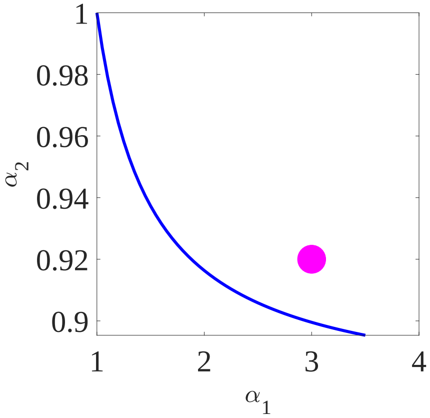
◇ Necessary and sufficient condition for determinacy:

$$\lim_{p\to\infty} \mathbb{E}_t A(s_t)\cdots A(s_{t+p}) = 0 \;\Leftrightarrow \rho(P \otimes A) < 1$$

Chapter 2: Some doubts

Farmer, Waggoner and Zha, AER (2010)

They construct **several solutions** in the determinacy region described by Davig and Leeper (2007)



Chapter 3: an alternative route

Farmer, Waggoner and Zha, JET (2009)

- Introduce a new stability concept(widely used in signal processing): Mean-square stability
- X_t is mean-square stable if, for any initial conditions, $\mathbb{E}_0(X_t)$ and $\mathbb{E}_0(X_t X_t')$ admit a limit.
- Necessary and sufficient condition for determinacy:

$$\lim_{p\to\infty}(\mathbb{E}_t\|A(s_t)\cdots A(s_{t+p})\|^2)^{1/2} = 0$$

NB: not a norm (so incompatible with implicit function theorem) but very convenient!

Chapter 4: Sophistication

Forward iteration method, Cho, RED (2016)

- For a general model

$$x_t = \mathbb{E}_t[A(s_t)x_{t+1}] + B(s_t)x_{t-1} + C(s_t)\epsilon_t,$$

it is shown that the solutions can be written under the form

$$x_t = \underbrace{[\Omega(s_t)x_{t-1} + \Gamma(s_t)\epsilon_t]}_{\text{Fundamental or MSV solution}} + \underbrace{w_t}_{\text{sunspot}},$$

$$w_t = F(s_t)\mathbb{E}_t[w_{t+1}],$$

where Ω, F and Γ are matrices built on A, B and C

- The process w can be written as (Farmer, Waggoner and Zha)

$$w_t = \Lambda(s_{t-1}, s_t)w_{t-1} + G(s_t)\eta_t, \quad \mathbb{E}_t[\eta_{t+1}] = 0$$

- Determinacy conditions for MSS in the general case (equivalence), but *assumptions on w*

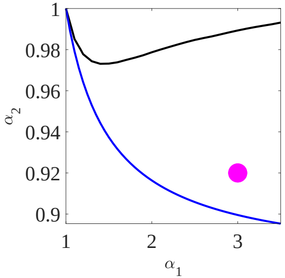
Chapter 5: Flashback

Boundedness again, Barthélemy and Marx, QE (2019)

- Necessary and sufficient conditions for determinacy

$$\lim_{p \to \infty} \mathbb{E}_t \| A(s_t) \cdots A(s_{t+p}) \| = 0$$

- Determinacy conditions for boundedness in the general case, built on the forward iteration method as Cho (2016)
- No equivalence in general, but *no assumptions on w*
- Puzzle: based on a very different approach compared to Cho (2016)



"When two forces unite, their efficiency double.", I. Newton

The process η such that $w_t = \Lambda(s_{t-1}, s_t)w_{t-1} + G(s_t)\eta_t$ satisfies the extra-assumption (standard in control theory).

Assumption. The sunspot shock η_t is bounded, mean-square stable and independent of w_{-1} and s_t.

- Key metric $\mu_n(M) = \lim_{k \to +\infty} \left(\mathbb{E}\|M(s_0)\cdots M(s_k)\|^n\right)^{1/nk}$
- Let Ω_1 be the matrix such that $\mu_2(\Omega_1')$ is minimal, $\tilde{\Omega}_1$ the matrix such that $\mu_\infty(\tilde{\Omega}_1')$ is minimal, determinacy is characterized by

Proposition

1. *There exists a unique bounded solution if and only if* $\mu_1(\tilde{F}_1) \leq 1$ *and* $\mu_\infty(\tilde{\Omega}_1') < 1$
2. *There exists a unique MSS solution if and only if* $\mu_2(F_1) \leq 1$ *and* $\mu_2(\Omega_1') < 1$

Epilogue: Typology

Complete classification of determinacy regions

$\mu_1(\tilde{F}_1) < \mu_2(F_1) \leq 1$	6. NSS(MSS) NSS(BDD)	4. DET(MSS) NSS(BDD)	1. DET(MSS) DET(BDD)
$\mu_1(\tilde{F}_1) \leq 1 < \mu_2(F_1)$		5. IND(MSS) NSS(BDD)	2. IND(MSS) DET(BDD)
$1 < \mu_1(\tilde{F}_1) < \mu_2(F_1)$			3. IND(MSS) IND(BDD)
	$1 \leq \mu_2(\Omega_1') < \mu_\infty(\tilde{\Omega}_1')$	$\mu_2(\Omega_1') < 1 \leq \mu_\infty(\tilde{\Omega}_1')$	$\mu_2(\Omega_1') < \mu_\infty(\tilde{\Omega}_1') < 1$

- ◇ Boundedness and asymptotic stability (convergence to the steady state in the absence of shocks) imply mean-square stability
- ◇ We have the following ranking

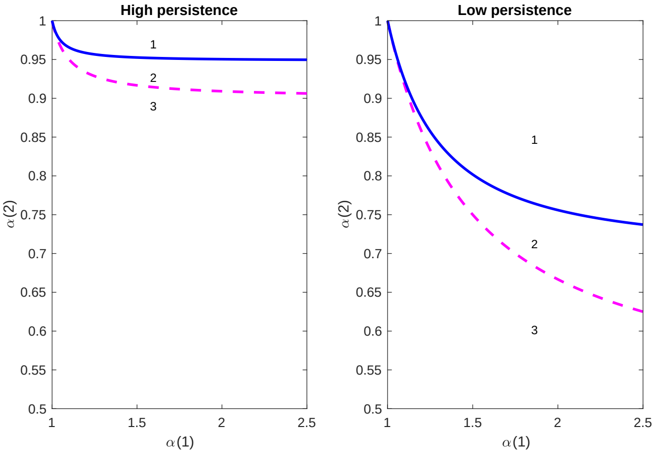
$$\mu_1(M) \leq \mu_2(M) \leq \mu_\infty(M)$$

Excipit

What does it change? It depends...

Application: Fisher equation + simplified regime switching Taylor rule

$$\mathbb{E}_t \pi_{t+1} = \alpha(s_t)\pi_t + \varepsilon_t^R$$



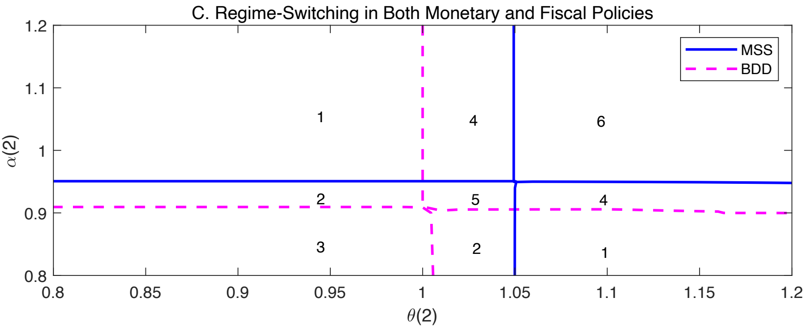
Excipit

General configuration, monetary and fiscal interactions

Application:

$$\alpha(s_t)\pi_t = \mathbb{E}_t\pi_{t+1} + \varepsilon_t^R,$$
$$b_t = \theta(s_t)b_{t-1} - c(s_t)\pi_t.$$

Regime 1: active monetary, passive fiscal.



Closing the book: Final word

This paper:

- ◇ Understanding of the two different stability concepts
- ◇ Quantitative assessment of determinacy for these two concepts
- ◇ Allows for use/comparison of these two concepts
- ◇ Codes are online here

Open issues:

- ◇ What is the relevant stability concept?
- ◇ Is the difference quantitatively important in more realistic models?